Representation stability for the cohomology of arrangements

Christin Bibby, University of Western Ontario

Motivating example: $\mathrm{Conf}_n(X)$

An *ordered configuration space* of n points on a space X is:

$$\mathrm{Conf}_n(X) := \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

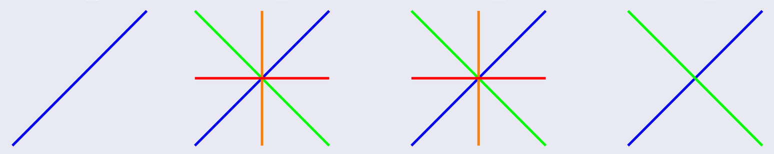
The symmetric group acts on X^n by permuting coordinates; this gives an action of S_n on $\mathrm{Conf}_n(X)$.

Note: $\mathrm{Conf}_n(\mathbb{C})$ is the complement to the union of hyperplanes associated to a type A_{n-1} root system, which are defined by $x_i = x_j$.

Hyperplane arrangements associated to root systems

Given a root system, one may consider the set of reflecting hyperplanes in $\mathbb{C}^n$. The corresponding Weyl group acts on the complement of their union.

Type:	A_1	B_2	C_2	D_2
Weyl group:	S_2	W_2	W_2	D_2



hyperplanes in $\mathbb{C}^n$ are defined by the equations:	A_1	B_2	C_2	D_2
	$x_i = x_j$	$x_i = x_j$	$x_i = x_j$	$x_i = x_j$
		$x_i = -x_j$	$x_i = -x_j$	$x_i = -x_j$
		$x_i = 0$	$2x_i = 0$	

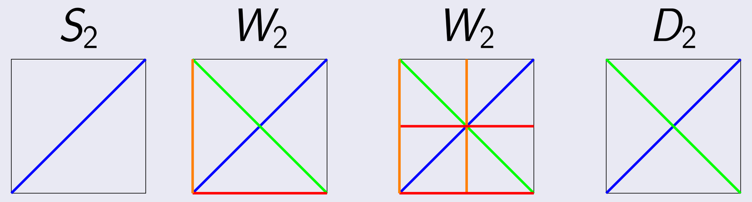
Question: What if we replace $\mathbb{C}$ with $\mathbb{C}^\times$ or a complex elliptic curve E?
We can make sense of these equations using the group operation. They will define (disjoint unions of) codimension-one subtori or abelian subvarieties, and the Weyl group will still act on the complement of their union.

Toric arrangements

Using the multiplicative group structure of $\mathbb{C}^\times$ to make sense of these equations, they define codimension-one subvarieties in $(\mathbb{C}^\times)^n$.

Here, we draw the arrangements in $(S^1)^2$ but imagine it's $(\mathbb{C}^\times)^2$:

Type:	A_1	B_2	C_2	D_2
Weyl group:	S_2	W_2	W_2	D_2



subtori in $(\mathbb{C}^\times)^n$ are defined by the equations:	A_1	B_2	C_2	D_2	
	$x_i = x_j$	$x_i = x_j$	$x_i = x_j$	$x_i = x_j$	H_{ij}
		$x_i = x_j^{-1}$	$x_i = x_j^{-1}$	$x_i = x_j^{-1}$	H'_{ij}
		$x_i = 1$	$x_i^2 = 1$		H_i^1, H_i^{-1}

What's new? The type B and C arrangements are different, since $x_i^2 = 1$ gives us two subvarieties. Moreover, $H_{12} \cap H'_{12} = \{(1,1), (-1,-1)\}$ has two connected components.
Note: The two-torsion points (1 and -1) play a key role here.

Elliptic arrangements

For a complex elliptic curve E, we consider codimension-one subvarieties of E^n.

Since a complex elliptic curve has *four* 2-torsion points, the solution to $2x_i = 0$, as well as the intersection $H_{ij} \cap H'_{ij}$ (where $x_i = x_j$ and $x_i = -x_j$), will each have four connected components (indexed by the 2-torsion points).

Note: Unfortunately, I don't have good pictures of elliptic arrangements.

Motivating example: $\mathrm{Conf}_n(\mathbb{C})$

Theorem. [Arnold,'69] The unordered configuration space on $\mathbb{C}$ is homologically stable. That is, for $n \gg 0$,

$$H_i(\mathrm{Conf}_n(\mathbb{C})/S_n; \mathbb{Q}) \cong H_i(\mathrm{Conf}_{n+1}(\mathbb{C})/S_{n+1}; \mathbb{Q})$$

Ordered configuration spaces don't have this property, but we can use the action of S_n to observe another type of stability. For example, for $n \geq 4$, we have the following decomposition as an S_n-representation:

$$H^1(\mathrm{Conf}_n(\mathbb{C}); \mathbb{Q}) = V_{(n)} \oplus V_{(n-1,1)} \oplus V_{(n-2,2)}$$

Note: Recall that irreducible representations of S_n are indexed by partitions of n.

Representation stability

Definition. [Church-Farb,'13] Let G_n denote either the symmetric group S_n or hyperoctahedral group $W_n = \mathbb{Z}_2 \wr S_n$. A sequence $\{V_n\}$ of G_n-representations with G_n-equivariant maps $\phi_n : V_n \to V_{n+1}$ is *uniformly representation stable* with stable range $n \geq N$ if for $n \geq N$...

- ϕ_n is injective,
- $G_{n+1} \cdot \phi_n(V_n) = V_{n+1}$, and
- $V_n = \oplus_\lambda V(\lambda)_n^{\oplus c_\lambda}$ where c_λ doesn't depend on n. (the multiplicities stabilize)

Note: For the symmetric groups, we consider $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash k$ such that $N \geq k + \lambda_1$, and $V(\lambda)_n$ denotes the irreducible representation of S_n corresponding to $\lambda[n] := (n-k, \lambda_1, \ldots, \lambda_\ell)$.
For the hyperoctahedral groups, irreducible representations are indexed by pairs of partitions, so for $\lambda = (\lambda^+, \lambda^-)$ with $\lambda^- \vdash k$, we take $V(\lambda)_n$ to be the representation of W_n indexed by the pair $(\lambda^+[n-k], \lambda^-)$.

Configuration spaces and hyperplane arrangements

Returning to $\mathrm{Conf}_n(\mathbb{C})$: We can restate our observation above that for $n \geq 4$,

$$H^1(\mathrm{Conf}_n(\mathbb{C}); \mathbb{Q}) = V(0) \oplus V(1) \oplus V(2)$$

The maps ϕ_n here are induced by $\mathrm{Conf}_{n+1}(\mathbb{C}) \to \mathrm{Conf}_n(\mathbb{C})$ which "forget the last point."

Theorem. [Church,'12] If X is a connected, orientable manifold, then for each i, $\{H^i(\mathrm{Conf}_n(X); \mathbb{Q})\}$ is uniformly representation stable with stable range $n \geq 4i$.

Theorem. [Wilson,'15] If $\{\mathcal{A}_n\}$ is a sequence of type A, B/C, or D arrangements in $\mathbb{C}^n$, with complements $M(\mathcal{A}_n)$, then for each i, $\{H^i(M(\mathcal{A}_n); \mathbb{Q})\}$ is uniformly representation stable with stable range $n \geq 4i$.

Note: Church used a Leray spectral sequence argument, and Wilson used so-called FI_W-modules. Combining these techniques, and understanding the combinatorics of our arrangements, gives our main theorem:

Main theorem

Let $\{\mathcal{A}_n\}$ be a sequence of toric or elliptic arrangements of type B, C, or D, with complements $M(\mathcal{A}_n)$. Then for each i, the sequence $\{H^i(M(\mathcal{A}_n); \mathbb{Q})\}$ of W_n-representations is uniformly representation stable with stable range $n \geq 4i$.

Some consequences:

- The orbit spaces $M(\mathcal{A}_n)/W_n$ are rationally homologically stable.
- For each i, $\dim H^i(M(\mathcal{A}_n); \mathbb{Q})$ is a polynomial in n.

Combinatorics: intersection poset

In type A, the set of intersections of subvarieties, partially ordered by reverse inclusion, is the partition lattice. More generally, if we take the set of connected components of intersections, partially ordered by reverse inclusion, we can describe it combinatorially using certain partitions, in a way that respects the action of the group:

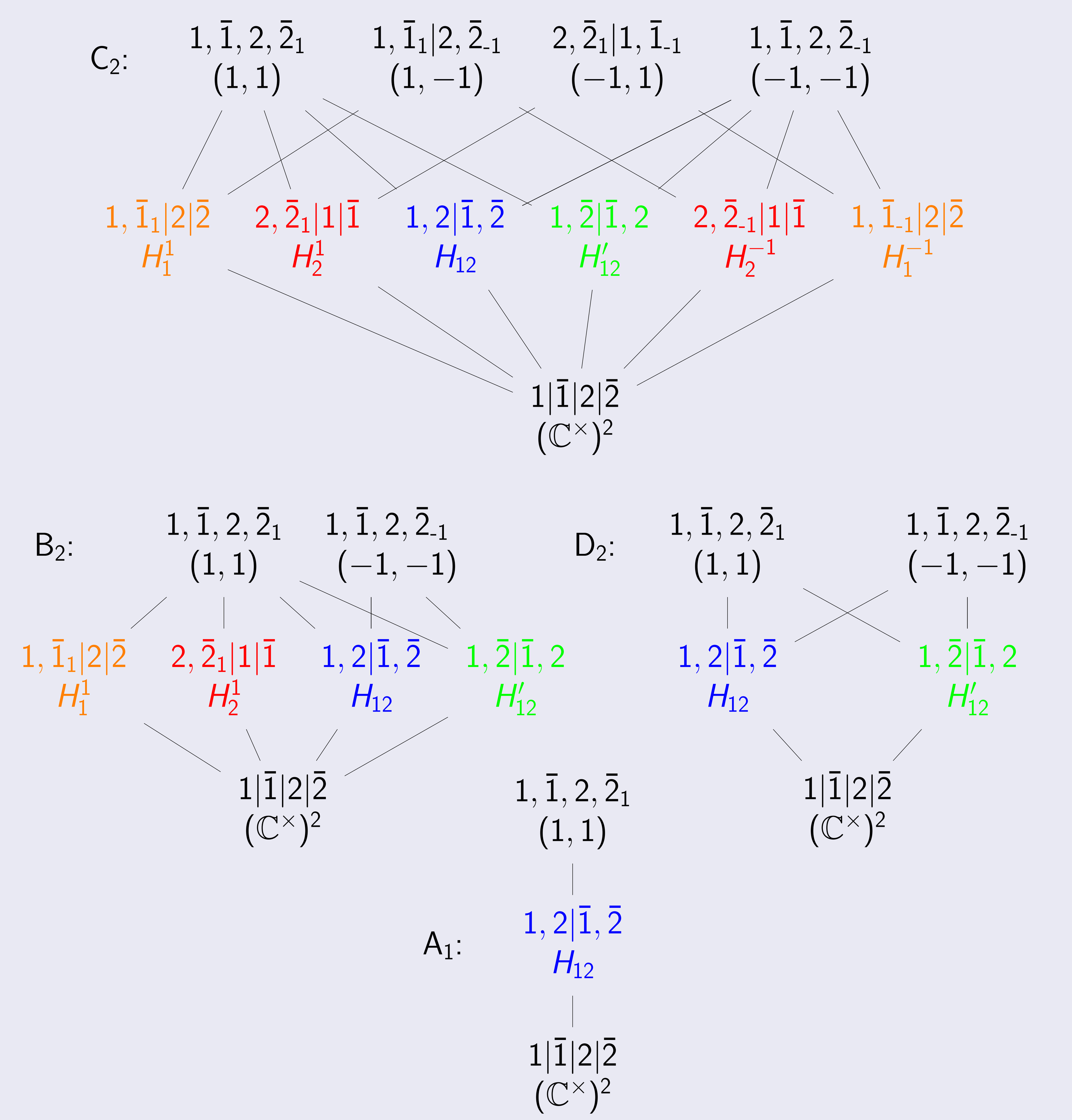
For type C_n arrangements in X^n ($X = \mathbb{C}, \mathbb{C}^\times, E$), connected components of intersections correspond to partitions π of $[n] = \{1, \bar{1}, \ldots, n, \bar{n}\}$, with each self-barred block labelled by a distinct 2-torsion point of X, such that for every $p \in \pi$ we have $\bar{p} \in \pi$.

For type B_n arrangements, we also require that if $\{i, \bar{i}\} \in \pi$ then it's labeled by the identity of X.
For type D_n arrangements, we require that there are no blocks of the form $\{i, \bar{i}\}$ in π.

Note: In the case of $\mathbb{C}$, this agrees with [Barcelo-Ihrig,'99].
For type B/C arrangements in $\mathbb{C}^n$ this is the Dowling lattice.

Note: Orbits are indexed by labelled partitions of n.
The labels and block sizes are preserved by the group action.

Examples of combinatorics in $(\mathbb{C}^\times)^2$



Some remaining questions

- What are the stable multiplicities?
  Note: Even the Betti numbers are hard to compute in the elliptic case.
- What about other complex reflection groups?
  Note: Complex multiplication on certain elliptic curves gives rise to interesting arrangements.