1. Introduction

A (complex) **hyperplane arrangement** is a finite set of hyperplanes in a complex vector space. An example of particular interest is the braid arrangement, which consists of the diagonal hyperplanes: for each $1 \leq i < j \leq n$, take the set of tuples $(x_1, \ldots, x_n)$ in $\mathbb{C}^n$ with $x_i = x_j$. The complement to this braid arrangement is the set of n-tuples of distinct complex numbers, which is called an **ordered configuration space** on $\mathbb{C}$. Arrangements and their complements are very rich objects with applications and connections to many branches of mathematics, going back to work of Arnol'd [Arn69], Brieskorn [Bri73], Orlik and Solomon [OS80], and Orlik and Terao [OT92].

More recently, with fundamental work by Looijenga [Loo93] and De Concini and Procesi [DCP05, DCP11], came the study of **toric arrangements**: collections of codimension-one subtori in a complex torus. For example, there is a toric analogue of the braid arrangement: for each $1 \leq i < j \leq n$, take the set of tuples $(x_1, \ldots, x_n)$ in $(\mathbb{C}^\times)^n$ with $x_i = x_j$. This analogy can be extended using a complex elliptic curve, which topologically is a real 2-torus, but along with $\mathbb{C}$ and $\mathbb{C}^\times$ they are the three types of dimension-one complex algebraic groups. This gives us **elliptic arrangements**, including an elliptic braid arrangement whose complement is an ordered configuration space on a complex elliptic curve.

These three flavors of arrangements have a similar local picture but a global picture affected by the ambient space, which makes each of them unique and interesting. One aspect which makes elliptic arrangements particularly exciting is the difference between an elliptic curve E and $\mathbb{C}$ or $\mathbb{C}^\times$. Topologically, E is a compact manifold; algebro-geometrically, E is a projective variety.

A general theme in arrangement theory is to try to understand the topology of the complement from the combinatorial structure. A goal of the theory of elliptic arrangements is to develop them as an elliptic analogue of hyperplane arrangements, and at the focal point of my research is the main question of arrangement theory:

To what extent does the combinatorics of the arrangement determine its topology?

The theories of hyperplane arrangements, toric arrangements, and configuration spaces are already rich, but there are compelling discoveries in the theory of elliptic arrangements that aren't seen there. This could open up new perspectives in and connections between various fields of mathematics (eg. combinatorics, topology, algebraic geometry). In particular, certain arrangements lend themselves well to the thriving field of representation stability, developed in Church-Farb [CF13] and Church-Ellenberg-Farb [CEF15].

2. Past Research Accomplishments

I have previous work on the topology of the complement to an elliptic arrangement, which includes describing a model for the cohomology, determining the rational homotopy theory of arrangements arising from chordal graphs, and proving that the cohomology of a sequence of arrangements associated to root systems stabilizes as a sequence of Weyl group representations. These things are outlined here, and more information can be found in my papers [Bib16a, Bib16b, BH16].

2.1. **DGA Models.**

In [Bib16a], I gave a method for computing the rational cohomology of the complement to the union of subvarieties in an elliptic arrangement. This generalizes a method used by Totaro in [Tot96] for computing the cohomology of a configuration space on a projective variety, using the Hodge structure on the Leray spectral sequence.

The main result obtained from using this method is a combinatorial presentation of a differential graded algebra (DGA) which is a (rational) model for the complement to a unimodular elliptic arrangement (that is, an arrangement where all intersections of subvarieties are connected). This presentation encodes both the geometry of the ambient space (its cohomology) and the combinatorics of the arrangement (with the Orlik-Solomon relation, coming from the cohomology of its linear counterpart):

Theorem 1. [Bib16a, Thm 4.1] *Let X be a complex abelian variety, $\mathcal{A} = \{H_1, \ldots, H_\ell\}$ a unimodular arrangement of codimension-one abelian subvarieties, and $M(\mathcal{A}) = X \setminus \cup_{i=1}^{\ell} H_i$ its complement. Define a differential graded algebra $A(\mathcal{A})$ as the quotient of the graded-commutative algebra $H^*(X;\mathbb{Q})[g_1, \ldots, g_\ell]$ by the ideal generated by*

(i) $\sum_{j=1}^{k} g_{i_1} \cdots \hat{g}_{i_j} \cdots g_{i_k}$ *whenever the intersection $H_{i_1} \cap \cdots \cap H_{i_k}$ has codimension less than k.*
(ii) $\alpha_i^*(x) g_i$ *where H_i is the kernel of the map $\alpha_i : X \to E_i$ for an elliptic curve E_i, and $x \in H^1(E_i;\mathbb{Q})$*

with differential d defined by $dg_i = [H_i] \in H^2(X;\mathbb{Q})$ and $dx = 0$ for $x \in H^*(X;\mathbb{Q})$. Then $A(\mathcal{A})$ is a model for $M(\mathcal{A})$, in the sense of rational homotopy theory.

Dupont [Dup15] also studied the more general case of the complement to a union of smooth hypersurfaces which intersect like hyperplanes in a smooth projective variety. He uses a similar but alternative method to that in [Bib16a], reaching similar results but not including the combinatorial presentation. In [Dup16], Dupont uses a decomposition of the Leray spectral sequence given in [Bib16a, Lemma 3.1], a generalization of Brieskorn's Lemma [Bri73], to show that all toric arrangements are formal. In [Suc16], Suciu uses this combinatorial model to study resonance varieties and formality of elliptic arrangements.

2.2. **Chordal Graphs and Rational Homotopy Theory.** With J. Hilburn in [BH16], we studied the model in Theorem 1 for arrangements arising from chordal graphs, which classify those with certain combinatorial properties (unimodular and supersolvable). In this case, techniques such as Gröbner basis theory and Koszul duality give us a combinatorial description for a Lie algebra which describes the rational homotopy theory for the complement to the chordal arrangement. Moreover, the results extend to studying chordal arrangements for higher genus projective curves:

Theorem 2. [BH16] *Let Γ be a chordal graph, C a complex projective curve of genus $g > 0$, $\mathcal{A}$ the associated arrangement in C^n, and $A(\mathcal{A})$ the Leray model (given in Theorem 1 when C is an elliptic curve). Then*

(i) $A(\mathcal{A})$ is a Koszul algebra and hence a Koszul DGA. Let $L = L(\mathcal{A})$ be the Lie algebra whose universal enveloping algebra $U(L)$ is the quadratic dual to $A(\mathcal{A})$.
(ii) $\overline{U(L)} \cong \overline{\mathbb{Q}[\pi_1(M(\mathcal{A}))]}$, where the completions are each with respect to the augmentation ideal.
(iii) The completion of L with respect to the filtration by bracket length is isomorphic to the Malcev Lie algebra of $\pi_1(M(\mathcal{A}))$.
(iv) The graded standard complex of L is the minimal model of $M(\mathcal{A})$.
(v) $M(\mathcal{A})$ is both a $K(\pi, 1)$ space and rationally a $K(\pi, 1)$ space.

The proof generalizes the methods used by Bezrukavnikov [Bez94] when studying Totaro's DGA for a configuration space on a complex projective curve. In [BMPP16], the authors use this Lie algebra to study (partial) formality of arrangements arising from a graph and positive genus complex projective curves, which they call partial configuration spaces.

2.3. **Representation Stability for Arrangements from Root Systems.** As mentioned in the introduction, there are certain arrangements, such as those arising from root systems, which enjoy representation stability. I have extended the results of Church [Chu12] for configuration spaces, and Wilson [Wil15, Wil14] for linear root system arrangements, to the toric and elliptic analogues of arrangements arising from root systems:

Theorem 3. [Bib16b, Thm 4.7] *Let $\{\mathcal{A}_n\}$ be a sequence of linear, toric, or elliptic arrangements of type A, B, C, or D, with complements $M(\mathcal{A}_n)$ and Weyl groups W_n. Then for each i, the sequence $\{H^i(M(\mathcal{A}_n);\mathbb{Q})\}$ of W_n-representations is uniformly representation stable with stable range $n \geq 4i$.*

A key ingredient to the proof is a combinatorial description of the poset of connected components of intersections of subvarieties in the arrangement, given using partitions with some extra structure [Bib16b, Thm 3.3]. This is a generalization of the type A case, where it is just the partition lattice. In the linear cases, the description agrees with that given by Barcelo and Ihrig [BI99], and in the type B/C linear case this is the Dowling lattice [Dow73].

3. Proposed Research: Objectives, Methods, Significance

There are still many questions on elliptic arrangements to be addressed, some of which are outlined here. The methods to be employed include combining and enhancing techniques used for the analogous hyperplane or toric arrangements with those used for the special case of ordered configuration spaces on an elliptic curve. These methods may also be extended to other "configuration-like" spaces.

3.1. **Computing Betti Numbers.** While the method for computing cohomology presented in [Bib16a] and the resulting DGA have proven to be useful, as outlined above, there are still open questions about the cohomology. In particular, for linear and toric arrangements, there is a combinatorial formula which determines the Poincaré and Hodge polynomials of the complement [DCP11, Moc12]. I seek an analogous statement for the elliptic case:

Problem 1. *Find a general (combinatorial) formula for the Poincaré and Hodge polynomials of the complement to an elliptic arrangement.*

The methods to solve this problem include the implementation into Macaulay2 of the method for computing cohomology via the DGA presentation in Theorem 1. Then using discrete Morse theory, as in [CGN15], I, with A. Thomas, will compute the Betti numbers (hence Poincaré polynomial) more generally, starting with ordered configuration spaces and arrangements arising from graphs.

While the linear and toric cases are known, this work would fill in a big gap: not even the case of an ordered configuration space on an elliptic curve is known. There is interest even in this special case, and some recent results on unordered configuration spaces [DCK16, Sch16] would follow from it.

Moreover, by construction of the DGA, it comes with a bigrading which encodes the weight filtration (in the sense of Hodge theory, see [Del75]), hence computing the Hodge polynomial. This can lead to answering more interesting problems; in particular on the rational homotopy theory. By the work of Dupont [Dup16], if the weight is determined to be pure up to a certain degree (which it has, in some cases), this implies partial formality.

3.2. **Homotopy Theory and the Fundamental Group.** While I have studied the rational homotopy theory, not much is known about the (non-rational) homotopy theory. In particular, it is known that chordal elliptic arrangements are $K(\pi, 1)$ spaces (in Theorem 2), but the fundamental group itself is not known:

Problem 2. *Determine the homotopy type, and in particular describe the fundamental group, of the complement to an elliptic arrangement.*

The methods to solve this problem would involve extending the notion of a Salvetti complex to elliptic arrangements. The Salvetti complex [Sal87] was one approach to the homotopy type (in particular, fundamental group) for a large class of hyperplane arrangements (called complexified), and it was extended to complexified toric arrangements by Moci and Settepenella [MS11] then d'Antonio and Delucchi [dD12].

Even solving this for a small class of arrangements would be significant; the best description of the fundamental group for the ordered configuration space is "a generalized pure braid group."

3.3. **Representation Stability and Posets.** Now going in a more algebraic direction, in view of recent work by Gadish [Gad16] on subspace arrangements and by Casto [Cas16] on complex reflection arrangements, I intend to extend Theorem 3 to other types of arrangements.

Problem 3. *Determine when a sequence of toric or elliptic arrangements (consisting of subvarieties, not necessarily of codimension-one) enjoys representation stability. In particular, consider toric and elliptic analogues of complex reflection arrangements. When representation stable, compute the stable multiplicities of the irreducible representations appearing in the cohomology.*

The methods would require a combinatorial understanding of the poset of connected components of subvarieties in an elliptic arrangement with a group action, building on my work for the case of root systems. The techniques used by Gadish, involving poset homology [Wac07], should then extend to this setting via a Leray spectral sequence. This work would contribute to the active and vibrant field of representation stability by extending techniques and by adding to the growing list of interesting examples.

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