A ONE-SIDED TAUBERIAN THEOREM FOR
THE BOREL SUMMABILITY METHOD

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ABSTRACT. We establish a quantitative version of Vijayaraghavan’s classical result
and use it to give a short proof of the known theorem that a real sequence \((s_n)\)
which is summable by the Borel method, and which satisfies the one-sided Tauberian
condition that \(\sqrt{n}(s_n - s_{n-1})\) is bounded below must be convergent.

1. Introduction and the main results. Suppose throughout that \((s_n)\) is a
sequence of real numbers, and that \(s_n = \sum_{k=0}^{n} a_k\). Let \(\alpha > 0\), let \(p_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^\alpha}\),
and let
\[
\sigma_\alpha(x) := \frac{1}{p_\alpha(x)} \sum_{k=0}^{\infty} \frac{s_k}{(k!)^\alpha} x^k
\]
for all \(x \in \mathbb{R}\).

Recall that the Borel summability method \(B\) is defined as follows:
\[
s_n \to s (B) \text{ if } \sum_{k=0}^{\infty} \frac{S_k}{k!} x^k \text{ is convergent for all } x \in \mathbb{R}, \text{ and } \sigma_1(x) \to s \text{ as } x \to \infty.
\]

For an inclusion result concerning the summability method based on \(\sigma_\alpha(x)\) see [3, p. 29]. Our aim is to give a short proof of the following well-known Tauberian
theorem for the Borel method [6, Theorem 241; and 9, 4].

Theorem 1. If \(s_n \to s (B)\), and if \(\sqrt{n}a_n \geq -c\) for some \(c \geq 0\) and all \(n \in \mathbb{N}\), then
\(s_n \to s\).

Our proof depends largely on the next result which is an improvement of Vija-
yarghavan’s theorem [6, Theorem 238; see also 8, 9] in that it specifies bounds in
its conclusion.

This research was supported in part by the Natural Sciences and Engineering Research Council
of Canada.

2000 Mathematics Subject Classification. 40G10; 40E05, 26A12.
Key words and phrases. Tauberian, Borel summability.

Typeset by A4S-TEX
Theorem 2. Let \( \alpha > 0 \), and suppose that

\[
(1) \liminf_{n \to \infty} \sqrt{n}a_n \geq -c_1 \text{ where } 0 \leq c_1 < \infty, \text{ and }
\]

\[
(2) \limsup_{n \to \infty} |\sigma_\alpha (n^\alpha \exp \left( \frac{\alpha}{2n} \right))| = c_2 < \infty.
\]

Then

\[
(3) \limsup_{n \to \infty} |s_n| \leq c_3 \left( c_2 + c_1 \left( \frac{2 \delta + \frac{1}{\delta \sqrt{2\pi \alpha}}}{\sqrt{2\pi \alpha}} \right) \right)
\]

for all \( \delta > \frac{\sqrt{2}}{\sqrt{\alpha \pi}} \) with \( c_3 = \left( 1 - \frac{2\sqrt{2}}{\delta \sqrt{2\pi \alpha}} \right)^{-1} \).

2. An auxiliary result. We require the following lemma for our proofs.

Lemma. Let \( \alpha > 0 \), \( \delta > 0 \), and let \( c_n(x) := \frac{1}{p_\alpha(x)} \cdot \frac{x^n}{(n!)^\alpha} \) for \( n \in \mathbb{N}_0 \). Moreover, suppose that \( M, N \in \mathbb{N} \), \( x := y^\alpha \) with

\[
y = y(n) := n \exp \left( \frac{1}{2n} \right), \quad M = M(n), \quad N = N(n) \text{ for } n \in \mathbb{N}, \text{ and define }
\]

\[
\Sigma_1 := \sum_{k=0}^{M} c_k(x), \quad \Sigma_2 := \sum_{k=N}^{\infty} c_k(x), \quad \text{and } \Sigma_3 := \sum_{k=N}^{\infty} \sum_{\nu=N}^{\infty} c_k(x) \sqrt{\nu}.
\]

Then

(i) \( \limsup_{M \to \infty} \Sigma_1 \leq \frac{1}{\delta \sqrt{2\pi \alpha}} \) whenever \( y \geq M + \delta \sqrt{M} \);

(ii) \( \limsup_{n \to \infty} \Sigma_2 \leq \frac{1}{\delta \sqrt{2\pi \alpha}} \) whenever \( N \geq y + \delta \sqrt{y} \); and

(iii) \( \limsup_{n \to \infty} \Sigma_3 \leq \frac{1}{\delta^2 \alpha \sqrt{2\pi \alpha}} \) whenever \( N \geq y + \delta \sqrt{y} \).

Proof. First, note that \( c_k(x) \) increases with \( k \) for \( 0 \leq k \leq y = x^{1/\alpha} \) and decreases for \( k \geq y \), and that, for \( 0 \leq k \leq m \leq y \),

\[
c_k(x) = c_m(x) \frac{(m(m-1) \cdots (k+1))^{\alpha}}{x^{m-k}} \\
\leq c_m(x) \left( \frac{m^{\alpha}}{x} \right)^{m-k} \leq c_m(x).
\]

Hence, for \( y \geq M + \delta \sqrt{M} \) with \( M \) large enough to ensure \( M \leq n \leq y \), we have that

\[
\Sigma_1 \leq c_M(x) \sum_{\nu=0}^{\infty} \left( \frac{M^{\alpha}}{x} \right) ^\nu \leq c_n(x) \left( 1 - \frac{M^{\alpha}}{y^\alpha} \right)^{-1}, \text{ where }
\]
(4) \[ \lim_{n \to \infty} c_n(x) \sqrt{n} = \sqrt{\frac{\alpha}{2\pi}}. \]

since \( x = n^\alpha \exp\left( \frac{\alpha}{2\pi} \right) \), by [2, Lemma 4.5.4; 5, p. 55; or 7]. Moreover

\[
\frac{1}{\sqrt{n}} \left(1 - \frac{M^\alpha}{y^\alpha} \right)^{-1} \leq \frac{1}{\sqrt{M}} \left(1 - \frac{M^\alpha}{(M + \delta \sqrt{M})^\alpha} \right)^{-1} \\
= \frac{1}{\sqrt{M}} \left(1 - (1 + \delta M^{-1/2})^{-\alpha} \right)^{-1} \to \frac{1}{\alpha \delta} \text{ as } M \to \infty,
\]

and this proves (i).

Next, we have that, for \( y = x^{1/\alpha} \leq m + 1 \leq k + 1, \)

\[
c_k(x) = c_m(x) \frac{x^{k-m}}{(m+1)(m+2) \cdots k)^\alpha} \leq c_m(x) \frac{x}{(m+1)^\alpha}^{k-m} \leq c_m(x).
\]

Hence, for \( N \geq y + \delta \sqrt{y} \), we have that

\[
\Sigma_2 \leq c_N(x) \sum_{\nu=0}^\infty \left( \frac{x}{N^\alpha} \right)^\nu \leq c_n(x) \left(1 - \frac{y^\alpha}{N^\alpha} \right)^{-1}, \text{ where}
\]

\[
\frac{1}{\sqrt{n}} \left(1 - \frac{y^\alpha}{N^\alpha} \right)^{-1} \leq \frac{1}{\sqrt{N}} \left(1 - \frac{y^\alpha}{(y + \delta \sqrt{y})^\alpha} \right)^{-1} \\
= \frac{1}{\sqrt{N}} \left(1 - (1 + \delta y^{-1/2})^{-\alpha} \right)^{-1} \to \frac{1}{\alpha \delta} \text{ as } y = n \exp\left( \frac{1}{2\pi n} \right) \to \infty,
\]

and this together with (4) implies (ii).

Finally, we see that, for \( N \geq y + \delta \sqrt{y} \),

\[
\Sigma_3 := \sum_{\nu=N}^{\infty} \sum_{k=\nu}^{\infty} c_k(x) \leq \sum_{\nu=N}^{\infty} c_\nu(x) \left(1 - \frac{x}{\nu^\alpha} \right)^{-1} \leq \frac{1}{\sqrt{N}} \left(1 - \frac{x}{N^\alpha} \right)^{-1} \sum_{\nu=N}^{\infty} c_\nu(x).
\]

Hence, by what we have shown before, we have that

\[
\limsup_{n \to \infty} \Sigma_3 \leq \frac{1}{\alpha \delta} \cdot \frac{1}{\delta \sqrt{2\pi \alpha}},
\]

which establishes (iii). \( \square \)

3. Proofs of the theorems.

Proof of Theorem 2. Let \( \alpha > 0 \) and \( \delta > \frac{2\sqrt{\alpha}}{\sqrt{\alpha \pi}} \). Given \( \varepsilon > 0 \), choose \( N_0 \in \mathbb{N} \) so large that

\[
a_n \geq -(c_1 + \varepsilon) \frac{1}{\sqrt{n}} \text{ for all } n \geq N_0, \text{ and}
\]
\( s_M > S_+(M) - \varepsilon \) and \(-s_N > S_-(N) - \varepsilon\)

for infinitely many integers \( M \) and \( N \) with \( M \geq N_0 \) and \( N \geq N_0 \), where

\[
S_+(m) := \max_{N_0 \leq k \leq m} s_k \quad \text{and} \quad S_-(m) := \max_{N_0 \leq k \leq m} (-s_k) \quad \text{for} \quad m \geq N_0.
\]

Note that the sequences \((S_+(m))\) and \((S_-(m))\) are non-decreasing, and that

\[
\max(S_+(m), S_-(m)) \geq |s_k| \quad \text{for} \quad N_0 \leq k \leq m.
\]

We consider two cases which exhaust all possibilities (cf. [6, pp. 308–311]).

**Case 1.** \( S_+(m) \geq S_-(m) \) for infinitely many integers \( m \).

Then there are infinitely many integers \( M \geq N_0 \) such that

\[
s_M > S_+(M) - \varepsilon \quad \text{and} \quad S_+(M) \geq S_-(M).
\]

We choose such an \( M \), and then integers \( n \) and \( N \) satisfying

\[
\begin{align*}
M + \delta \sqrt{M} &\leq y := n \exp \left( \frac{1}{2m} \right) < M + \delta \sqrt{M} + 2, \\
y + \delta \sqrt{y} &\leq N < y + \delta \sqrt{y} + 2,
\end{align*}
\]

and we put \( x := y^n \). Then \( \sqrt{N} \leq \sqrt{M} + \delta + \frac{2}{\sqrt{M}} \), because

\[
N < \left( \sqrt{y} + \frac{\delta}{2} + \frac{1}{\sqrt{y}} \right)^2 \quad \text{and} \quad y < \left( \sqrt{M} + \frac{\delta}{2} + \frac{1}{\sqrt{M}} \right)^2.
\]

We split \( \sigma_\alpha(x) \) as follows:

\[
\sigma_\alpha(x) := \sum_{\nu=1}^{4} \tau_\nu(x), \quad \text{where}
\]

\[
\begin{align*}
\tau_1(x) &:= \sum_{k=0}^{N_0} s_k c_k(x), \\
\tau_2(x) &:= \sum_{k=N_0+1}^{M} s_k c_k(x), \\
\tau_3(x) &:= \sum_{k=M+1}^{\infty} s_M c_k(x), \\
\tau_4(x) &:= \sum_{k=M+1}^{\infty} (s_k - s_M) c_k(x).
\end{align*}
\]

We see immediately that \( \tau_1(x) \to 0 \) as \( M \to \infty \).

In what follows we use the notation of the Lemma. By (5), we have that \(-s_k \leq S_-(k) \leq S_-(M) \leq S_+(M) < s_M + \varepsilon\) for \( 0 \leq k \leq M \), and hence that

\[
\tau_2(x) \geq -(s_M + \varepsilon) \Sigma_1.
\]

Next, we observe that

\[
\tau_3(x) = s_M (1 - \Sigma_1).
\]
Finally, we see that
\[
\tau_4(x) = \sum_{k=M+1}^{\infty} \sum_{\nu=M+1}^{k} a_\nu c_k(x) \geq -(c_1 + \varepsilon) \sum_{k=M+1}^{\infty} \sum_{\nu=M+1}^{k} \frac{c_k(x)}{\sqrt{\nu}} \\
= -(c_1 + \varepsilon) \left( \tau_{4,1}(x) + \tau_{4,2}(x) \right),
\]
where
\[
\tau_{4,1}(x) := \sum_{k=M+1}^{\infty} \min(k, N) \sum_{\nu=M+1}^{\min(k, N)} \frac{c_k(x)}{\sqrt{\nu}} \leq \sum_{k=M+1}^{\infty} c_k(x) \int_{M}^{N} \frac{dt}{\sqrt{t}}
\]
\[
= 2(\sqrt{N} - \sqrt{M}) \sum_{k=M+1}^{\infty} c_k(x) \leq 2 \left( \delta + \frac{2}{\sqrt{M}} \right),
\]
and
\[
\tau_{4,2}(x) := \sum_{k=N+1}^{\infty} \sum_{\nu=N+1}^{k} \frac{c_k(x)}{\sqrt{\nu}} \leq \Sigma_3.
\]

Collecting the above results, we see that
\[
\sigma_\alpha(x) \geq \tau_1(x) + s_M(1 - 2\Sigma_1) - \varepsilon \Sigma_1 - (c_1 + \varepsilon) \left( 2\delta + \frac{4}{\sqrt{M}} + \Sigma_3 \right).
\]

Since \( \varepsilon \) is an arbitrary positive number, and
\[
\liminf_{M \to \infty} s_M + \varepsilon \geq \lim_{m \to \infty} S_+(m) = \lim_{m \to \infty} \max \left( S_+(m), S_-(m) \right) \geq \limsup_{m \to \infty} |s_m|,
\]

it follows from (7) that
\[
\liminf_{M \to \infty} s_M(1 - 2\limsup_{M \to \infty} \Sigma_1) \leq \limsup_{M \to \infty} \sigma_\alpha(x) + c_1 \left( 2\delta + \limsup_{M \to \infty} \Sigma_3 \right).
\]

and hence, by the Lemma, that
\[
\limsup_{m \to \infty} |s_m| \left( 1 - \frac{\sqrt{2}}{\delta \sqrt{\alpha \pi}} \right) \leq c_2 + c_1 \left( 2\delta + \frac{1}{\delta^2 \alpha \sqrt{2 \pi \alpha}} \right),
\]
which yields assertion (3) in Case 1.

Case 2. \( S_+(m) < S_-(m) \) for all \( m \geq N_1 \geq N_0 \).

We choose integers \( M, n, N \) to satisfy (6) as in Case 1. In addition, we choose \( N \geq N_1 \) such that \(-s_N > S_-(N) - \varepsilon\), which is evidently possible for large \( N \). We now split \( \sigma_\alpha(x) \) as follows:
\[
\sigma_\alpha(x) := \sum_{\nu=1}^{6} \tau_\nu(x),
\]
where
\[
\tau_1(x) := \sum_{k=0}^{N_1} s_k c_k(x), \quad \tau_2(x) := \sum_{k=N_1+1}^{M} s_k c_k(x),
\]
and
\[
\tau_3(x) := \sum_{k=M+1}^{\infty} \sum_{\nu=M+1}^{k} a_\nu c_k(x) \geq -(c_1 + \varepsilon) \sum_{k=M+1}^{\infty} \sum_{\nu=M+1}^{k} \frac{c_k(x)}{\sqrt{\nu}} \\
= -(c_1 + \varepsilon) \left( \tau_{3,1}(x) + \tau_{3,2}(x) \right),
\]
where
\[
\tau_{3,1}(x) := \sum_{k=M+1}^{\infty} \min(k, N) \sum_{\nu=M+1}^{\min(k, N)} \frac{c_k(x)}{\sqrt{\nu}} \leq \sum_{k=M+1}^{\infty} c_k(x) \int_{M}^{N} \frac{dt}{\sqrt{t}}
\]
\[
= 2(\sqrt{N} - \sqrt{M}) \sum_{k=M+1}^{\infty} c_k(x) \leq 2 \left( \delta + \frac{2}{\sqrt{M}} \right),
\]
and
\[
\tau_{3,2}(x) := \sum_{k=N+1}^{\infty} \sum_{\nu=N+1}^{k} \frac{c_k(x)}{\sqrt{\nu}} \leq \Sigma_3.
\]

Collecting the above results, we see that
\[
\sigma_\alpha(x) \geq \tau_1(x) + s_M(1 - 2\Sigma_1) - \varepsilon \Sigma_1 - (c_1 + \varepsilon) \left( 2\delta + \frac{4}{\sqrt{M}} + \Sigma_3 \right).
\]

Since \( \varepsilon \) is an arbitrary positive number, and
\[
\liminf_{M \to \infty} s_M + \varepsilon \geq \lim_{m \to \infty} S_+(m) = \lim_{m \to \infty} \max \left( S_+(m), S_-(m) \right) \geq \limsup_{m \to \infty} |s_m|,
\]

it follows from (7) that
\[
\liminf_{M \to \infty} s_M(1 - 2\limsup_{M \to \infty} \Sigma_1) \leq \limsup_{M \to \infty} \sigma_\alpha(x) + c_1 \left( 2\delta + \limsup_{M \to \infty} \Sigma_3 \right).
\]

and hence, by the Lemma, that
\[
\limsup_{m \to \infty} |s_m| \left( 1 - \frac{\sqrt{2}}{\delta \sqrt{\alpha \pi}} \right) \leq c_2 + c_1 \left( 2\delta + \frac{1}{\delta^2 \alpha \sqrt{2 \pi \alpha}} \right),
\]
which yields assertion (3) in Case 1.
Finally, we observe that, for \( k \) small enough, \( \varepsilon \) Since
\[
N \to \infty
\]
We see immediately that
\[
\tau_2(x) \leq (-s_N + \varepsilon)\Sigma_1.
\]
Next, we observe that
\[
\tau_3(x) = s_N(1 - \Sigma_1).
\]
Further, we see that
\[
\tau_4(x) = \sum_{k=M+1}^{N-1} \sum_{\nu=k+1}^{N} (-a_\nu)c_k(x) \leq (c_1 + \varepsilon) \sum_{k=M+1}^{N-1} \sum_{\nu=M+1}^{k} \frac{c_k(x)}{\sqrt{\nu}}
\]
\[
\leq (c_1 + \varepsilon) \sum_{k=M+1}^{\infty} c_k(x) \int_{\nu=M}^{N} \frac{dt}{\sqrt{t}} = 2(c_1 + \varepsilon)(\sqrt{N} - \sqrt{M}) \sum_{k=M+1}^{\infty} c_k(x)
\]
\[
\leq 2(c_1 + \varepsilon) \left( \delta + \frac{2}{\sqrt{M}} \right),
\]
and that
\[
\tau_5(x) = -2s_N\Sigma_2.
\]
Finally, we observe that, for \( k \geq N \geq N_1 \geq N_0 \), either \( s_k \leq S_+(k) < S_-(k) = \max_{N_0 \leq \nu \leq k} (-s_\nu) = -s_m \) for some \( m \in (N,k) \), in which case we have that
\[
s_k + s_N \leq s_N - s_m = \sum_{\nu=N+1}^{m} (-a_\nu) \leq (c_1 + \varepsilon) \sum_{\nu=N+1}^{k} \frac{1}{\sqrt{\nu}},
\]
or \( s_k \leq S_-(N) < -s_N + \varepsilon \). It follows that
\[
\tau_6(x) \leq (c_1 + \varepsilon) \sum_{k=N}^{\infty} \sum_{\nu=N}^{k} \frac{c_k(x)}{\sqrt{\nu}} + \varepsilon\Sigma_2 = (c_1 + \varepsilon)\Sigma_3 + \varepsilon\Sigma_2.
\]
Collecting the above results, we see that
\[
(8) \quad \sigma_\alpha(x) \leq \tau_1(x) + s_N(1 - 2\Sigma_1 - 2\Sigma_2) + 2(c_1 + \varepsilon) \left( \delta + \frac{2}{\sqrt{M}} \right) + (c_1 + \varepsilon)\Sigma_3 + \varepsilon.
\]
Since \( \varepsilon \) is an arbitrary positive number, and
\[
\liminf_{N \to \infty} (-s_N) + \varepsilon \geq \lim_{m \to \infty} S_-(m) = \lim_{m \to \infty} \max_{m \to \infty} (S_+(m), S_-(m)) \geq \limsup_{m \to \infty} |s_m|,
\]
it follows from (8) that
\[
\liminf_{N \to \infty} (-s_N) \left( 1 - 2 \limsup_{N \to \infty} \sum_1 - 2 \limsup_{N \to \infty} \sum_2 \right) \\
\leq \limsup_{N \to \infty} \left( - \sigma(x) \right) + c_1 \left( 2 \delta + \limsup_{N \to \infty} \sum_3 \right),
\]
and hence, by the Lemma, that
\[
\limsup_{m \to \infty} |s_m| \left( 1 - \frac{2 \sqrt{2}}{\delta \sqrt{\alpha \pi}} \right) \leq c_2 + c_1 \left( 2 \delta + \frac{1}{\delta^2 \alpha \sqrt{2 \pi \alpha}} \right),
\]
which yields assertion (3) in Case 2. \(\square\)

We now discuss consequences of Theorem 2. The corresponding two-sided result is [2, Lemma 4.5.5; 7, Lemma 5], and the arguments from now on are much the same as those in the references.

**Proposition 1.** (Cf. the o-Tauberian theorem [2, Corollary 4.3.8]) Suppose that \(s_n \to s(B)\), and that \(\liminf_{n \to \infty} \sqrt{n} a_n \geq 0\), then \(s_n \to s\).

**Proof.** We may assume without loss of generality that \(s = 0\), so that \(\lim_{x \to \infty} \sigma_1(x) = 0\). Then Theorem 2 can be applied with \(c_1 = c_2 = 0, \alpha = 1\), and any \(\delta > \frac{2 \sqrt{2}}{\sqrt{\pi}}\), to yield \(\limsup_{n \to \infty} |s_n| = 0\), i.e., \(s_n \to 0\). \(\square\)

Observe that we did not need the full proof of (4) in [2] or [7] which involved asymptotic approximations valid for all \(\alpha > 0\). For the case \(\alpha = 1\), only Stirling’s formula is used.

**Proposition 2.** (Boundedness) Suppose that \(\sigma_\alpha(x)\) is bounded as \(x \to \infty\) for some \(\alpha > 0\), and that condition (1) of Theorem 2 holds. Then the sequence \((s_n)\) is bounded.

**Proof.** The result follows from Theorem 2 with any \(\delta > \frac{2 \sqrt{2}}{\sqrt{\alpha \pi}}\). \(\square\)

**Proof of Theorem 1.** We may again assume without loss of generality that \(s = 0\), i.e., that \(s_n \to 0 (B)\). Then, by Proposition 2, \((s_n)\) is bounded, and it follows from [2, Theorem 4.5.2 and Proof of Theorem 4.5.1 on p. 200] (see also [7] and [1]) that \(\sigma_\alpha (n^\alpha \exp \left( \frac{2 \sqrt{n}}{\alpha} \right)) \to 0 \) as \(n \to \infty\) for all \(\alpha > 0\). Hence, by Theorem 2 with \(c_2 = 0\),
\[
\limsup_{n \to \infty} |s_n| \leq \left( 1 - \frac{2 \sqrt{2}}{\delta \sqrt{\alpha \pi}} \right)^{-1} c_1 \left( 2 \delta + \frac{1}{\delta^2 \alpha \sqrt{2 \pi \alpha}} \right)
\]
for all \(\alpha > 0\) and \(\delta > \frac{2 \sqrt{2}}{\sqrt{\pi}}\). Letting \(\delta \to 0, \alpha \to \infty\), subject to \(\delta \sqrt{\alpha} \to \infty\), we obtain the required conclusion that \(s_n \to 0\). \(\square\)
References


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