1. Inversions

If $\sigma$ is a permutation of $[n]$, say that a pair $(i, j)$ is an inversion of $\sigma$ if $1 \leq i < j \leq n$ and $\sigma_i > \sigma_j$: that is, if the number in position $i$ exceeds the number in position $j$.

**Example 1.1.** The permutation $\sigma = [3142]$ has three inversions: $(1, 2)$, $(1, 4)$, and $(3, 4)$.

The length of a permutation is, by definition, the number of inversions in $\sigma$, written $l(\sigma)$.

We can also count the number of inversions for which the left-hand position is fixed: for $1 \leq i \leq n$, let $m_i$ be the number of $j$ with $i < j \leq n$ for which $(i, j)$ is an inversion. Clearly $m_n = 0$.

**Example 1.2.**
- For $\sigma = [3142]$, $m_1 = 2$, $m_2 = 0$, $m_3 = 1$ and $m_4 = 0$.
- For $\sigma = [123\cdots n]$, clearly $m_i = 0$ for all $i$, $1 \leq i \leq n$.
- For $\sigma = [n, n-1\cdots 321]$, we have $m_i = n - i$ for $1 \leq i \leq n$.

It is not hard to check that $0 \leq m_i \leq n - i$ for each $i$, $1 \leq i \leq n$. In fact, every inversion sequence $(m_1, \ldots, m_n)$ is possible:

**Theorem 1.3.** For a fixed integer $n \geq 1$, suppose $(m_1, \ldots, m_n)$ are integers satisfying $0 \leq m_i \leq n - i$ for $1 \leq i \leq n$. Then there is a unique permutation $\sigma$ of $[n]$ for which $(m_1, \ldots, m_n)$ is its inversion sequence.

**Proof.** We can check that there are exactly $n!$ such sequences, so it is enough to show that there is some permutation $\sigma$ having given numbers $(m_1, \ldots, m_n)$ as its inversion sequence. The uniqueness follows by counting.

Given $(m_1, \ldots, m_n)$, make a permutation recursively by letting $S = [n]$, $i = 1$, and then

- If $S = \emptyset$, stop. Otherwise,
- Let $\sigma_i$ be the $m_i + 1$st element of the set $S$, written in increasing order.
- Delete the number $\sigma_i$ from the set $S$.
- Increase $i$ by one and repeat.

(The reader needs to check that the resulting $\sigma$ really is a permutation, and $(m_1, \ldots, m_n)$ is its inversion sequence.)

2. Generating functions

With this in mind, we can make a generating function for the permutations of $[n]$ of a given length. By Theorem 1.3, the number of permutations of length $d$ is just the number of solutions to

$$m_1 + m_2 + \cdots + m_n = d,$$
where each \( m_i \) is an integer and \( 0 \leq m_i \leq n - i \). By the multiplication principle for ordinary generating functions, this is the coefficient of \( t^d \) in the product
\[
(1 + t + \cdots + t^{n-1})(1 + t + \cdots + t^{n-2})\cdots (1 + t + t^2)(1 + t)(1).
\]
That is, if we let \( S_n \) be the set of all permutations of \([n]\), we get the generating function
\[
\sum_{\sigma \in S_n} t^{l(\sigma)} = \prod_{i=1}^{n} \sum_{j=0}^{n-i-1} t^j
= \prod_{i=1}^{n} \frac{1 - t^{n-i}}{1 - t}
= \frac{1}{(1 - t)^n} \prod_{i=1}^{n} (1 - t^i).
\]
Finally, notice that this is really a polynomial, and that if you evaluate either side at \( t = 1 \), you get \( n! \).

**Example 2.1.** For \( n = 3 \), this is \((1 + t + t^2)(1 + t) = 1 + 2t + 2t^2 + t^3\). For \( n = 4 \), we get the polynomial
\[
1 + 3t + 5t^2 + 6t^3 + 5t^4 + 3t^5 + t^6.
\]