Efficient Algorithms for Computing the Betti Numbers of Semi-algebraic Sets

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Semi-algebraic Sets

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Basic Properties of Semi-algebraic Sets

- Closed under union, intersection, complementation and linear projection.

They satisfy various finiteness properties. Finite number of connected components, finite Betti numbers etc. In particular, compact semi-algebraic sets are finitely triangulable.

Uniform bounds on their topological complexity: number of connected components, Betti numbers etc.

Measuring complexity in terms of:

- Number of polynomials (controls the combinatorial complexity).
- Degree bound (controls the algebraic complexity).
- Dimension of the ambient space.
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Some motivations ...

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- Studying certain questions in quantitative real algebraic geometry. For instance, existence of single exponential sized triangulations.
- Recent work in complexity theory (Cucker, Buergisser) on the real version of counting complexity classes.
- Some ideas may be useful in designing algorithms for computing homology groups in other contexts.
Cylindrical Algebraic Decomposition

Effective method of decomposing semi-algebraic sets.
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Complexity is double exponential in the dimension.

\((O(nd))^{2^k}\)
Classical Result

**Theorem 1** (Oleinik and Petrovsky, Thom, Milnor) Let $S \subset \mathbb{R}^k$ be the set defined by the conjunction of $n$ inequalities,

\[ P_1 \geq 0, \ldots, P_n \geq 0, \]

\[ P_i \in [X_1, \ldots, X_k], \deg(P_i) \leq d, 1 \leq i \leq n. \]
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$$P_i \in [X_1, \ldots, X_k], \ deg(P_i) \leq d, 1 \leq i \leq n.$$

Then,

$$\sum_i b_i(S) \leq nd(2nd - 1)^{k-1} = O(nd)^k.$$
Tightness

The above bound is actually quite tight. Example: Let

\[ P_i = L_{i,1}^2 \cdots L_{i,[d/2]}^2 - \epsilon, \]

where the \( L_{ij} \)'s are generic linear polynomials and \( \epsilon > 0 \) and sufficiently small. The set \( S \) defined by \( P_1 \geq 0, \ldots, P_n \geq 0 \) has \( \Omega(n^d)^k \) connected components and hence \( b_0(S) = \Omega(n^d)^k \).
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What about the higher Betti Numbers?

- Cannot construct examples such that $b_i(S) = \Omega(nd)^k$ for $i > 0$. 
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- Cannot construct examples such that \( b_i(S) = \Omega(nd)^k \) for \( i > 0 \).

- The technique used for proving the above result does not help: Replace the semi-algebraic set \( S \) by another set bounded by a smooth algebraic hypersurface of degree \( nd \) having the same homotopy type as \( S \). Then bound the Betti numbers of this hypersurface using Morse theory and the Bezout bound on the number of solutions of a system of polynomial equations.
Picture Proof of Thom-Milnor Bound

Connected component of $S$
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Connected component of $S$

$Z(Q_t)$
Complexity of Algorithms

- Double exponential vs single exponential vs polynomial time.

Problems that can be solved in single exponential time:
- Testing emptiness,
- Deciding connectivity,
- Computing descriptions of the connected components,
- Computing the Euler-Poincaré characteristic of a given semi-algebraic set.

Problems for which no single exponential time algorithm is known:
- Computing the higher Betti numbers,
- Computing semi-algebraic triangulations,
- Computing semi-algebraic stratifications.
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- Problems for which no single exponential time algorithm is known: Computing the higher Betti numbers, computing semi-algebraic triangulations, or semi-algebraic stratifications.
New Results

- Single exponential time algorithm for computing the first Betti number of semi-algebraic sets (with R. Pollack, M-F. Roy).
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- Polynomial time algorithm to compute the top Betti numbers of semi-algebraic sets defined by quadratic inequalities.
Another approach

Let $A_1, \ldots, A_n$ be subcomplexes of a finite simplicial complex $A$ such that $A = A_1 \cup \cdots \cup A_n$. Let $C^i(A)$ denote the $i$-vector space of $i$ co-chains of $A$, and $C^*(A) = \bigoplus_i C^i(A)$. 
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We will denote by $A_{\alpha_0, \ldots, \alpha_p}$ the subcomplex $A_{\alpha_0} \cap \cdots \cap A_{\alpha_p}$.
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The following sequence of homomorphisms is exact.

$$0 \longrightarrow C^*(A) \xrightarrow{r} \prod_{\alpha_0} C^*(A_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0 < \alpha_1} C^*(A_{\alpha_0, \alpha_1})$$

$$\cdots \xrightarrow{\delta} \prod_{\alpha_0 < \cdots < \alpha_p} C^*(A_{\alpha_0, \ldots, \alpha_p}) \cdots \xrightarrow{\delta} \prod_{\alpha_0 < \cdots < \alpha_{p+1}} C^*(A_{\alpha_0, \ldots, \alpha_{p+1}}) \cdots \xrightarrow{\delta} \cdots$$
Mayer-Vietoris Double Complex

We now consider the following bigraded double complex $\mathcal{M}^{p,q}$, with a total differential $D = \delta + (-1)^p d$, where

\[
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\uparrow d & \uparrow d & \uparrow d & \uparrow d & \uparrow d & \uparrow d & \uparrow d \\
0 & \rightarrow & \Pi_{\alpha_0} C^3(A_{\alpha_0}) & \rightarrow & \Pi_{\alpha_0 < \alpha_1} C^3(A_{\alpha_0, \alpha_1}) & \rightarrow & \Pi_{\alpha_0 < \alpha_1 < \alpha_2} C^3(A_{\alpha_0, \alpha_1, \alpha_2}) \\
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0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0 \\
\end{array}
\]
Double Complex

\[
\begin{array}{ccc}
\cdots & d & \cdots \\
C^{0,2} & \delta & C^{1,2} & \delta & C^{2,2} & \delta & \cdots \\
\uparrow d & \uparrow & \uparrow d & \uparrow d & \uparrow d & \uparrow d & \\
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The Associated Total Complex
Spectral Sequences of a Double Complex

A sequence of vector spaces progressively approximating the homology of the total complex. More precisely,
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- a sequence of bi-graded vector spaces and differentials

$$ (E_r, d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}) $$
A sequence of vector spaces progressively approximating the homology of the total complex. More precisely,

- a sequence of bi-graded vector spaces and differentials $(E_r, d_r : E_r^{p,q} \rightarrow E_r^{p+r,q-r+1})$,

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  \[ (E_r, d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}) , \]
- \[ E_{r+1} = H(E_r, d_r) , \]
- \[ E_\infty = H^*(\text{Associated Total Complex}) . \]
Spectral Sequence

\[ p + q = i \quad p + q = i + 1 \]
Two Spectral Sequences

There are two spectral sequences associated with $\mathcal{M}^{p,q}$ both converging to $H^*_D(\mathcal{M})$. The first terms of these are:
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$$E'_1 = H_d(\mathcal{M}), \quad E'_2 = H_\delta H_d(\mathcal{M})$$
$E_1$

\[ C^3(A) \ 0 \ 0 \]

\[ C^2(A) \ 0 \ 0 \]

\[ C^1(A) \ 0 \ 0 \]

\[ C^0(A) \ 0 \ 0 \]
$$E_2$$

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Inequality I

Let $A$ be a finite simplicial complex and $A_1, \ldots, A_n$ sub-complexes of $A$ such that $A = A_1 \cup \cdots \cup A_n$. Then for every $i \geq 0,$

$$b_i(A) \leq \sum_{j=1}^{i+1} \sum_{J \subset \{1, \ldots, n\}, \#(J) = j} b_{i-j+1}(A_J),$$

where $A_J = \cap_{j \in J} A_j$. 
Inequality II

Let be a real closed field and $V \subset^k$ be the set defined by the conjunction of $\ell$ inequalities,

$$P_1 \geq 0, \ldots, P_\ell \geq 0, P_i \in R[X_1, \ldots, X_k],$$

$$\deg(P_i) \leq d, 1 \leq i \leq \ell,$$

contained in a variety $Z(Q)$ of real dimension $k'$ with $\deg(Q) \leq d$.

Then, for all $i$, $0 \leq i \leq k'$,

$$b_i(V) \leq (3^\ell - 1)d(2d - 1)^{k-1}.$$
Graded Bounds

**Theorem 2** (B, 2001) Let $S \subseteq R^k$ (resp. $T \subseteq R^k$) be the set defined by the conjunction (resp. disjunction) of $n$ inequalities,

$$P_1 \geq 0, \ldots, P_n \geq 0, P_i \in R[X_1, \ldots, X_k], \text{ deg}(P_i) \leq d, 1 \leq i \leq n,$$

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restricted to a variety $Z(Q)$ of real dimension $k'$ with $\text{deg}(Q) \leq d$. Then,

$$b_i(S) \leq \sum_{j=0}^{k'-i} \binom{n}{j} 2^{j+1} d(2d - 1)^{k-1} = \binom{n}{k' - i} O(d)^k,$$

$$b_i(T) \leq \sum_{j=0}^{i+1} \binom{n}{j} 3^j d(2d - 1)^{k-1} = \binom{n}{i + 1} O(d)^k.$$
Sets defined by Quadratic Inequalities

**Theorem 3** (B, 2001) Let \( \ell \) be any fixed number and let \( S \subset R^k \) be defined by \( P_1 \geq 0, \ldots, P_n \geq 0 \) with \( \deg(P_i) \leq 2 \). Then,

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b_{k-\ell}(S) \leq \binom{n}{\ell} k^{O(\ell)}.
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**Sets defined by Quadratic Inequalities**

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This bound is *polynomial in the dimension* $k$ unlike the O-P-T-M bound which was *single exponential in* $k$. Notice also that the lowest Betti numbers of $S$ cannot be polynomially bounded. Example: $S$ defined by

$$X_1(X_1 - 1) \geq 0, \ldots, X_k(X_k - 1) \geq 0.$$

Clearly, in this case, $b_0(S) = 2^k$. 
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Algorithm for computing $b_1$

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Recall that $E'_1$ is:

\[\begin{align*}
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\end{align*}
\]
Suppose that we can compute an acyclic covering in single exponential time. Then, all $H^i(A_j) = 0, i > 0$. Then, $H_1(A)$ is isomorphic to the middle homology of,

$$
\prod_{\alpha_0} H^0(A_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0 < \alpha_1} H^0(A_{\alpha_0, \alpha_1}) \xrightarrow{\delta} \prod_{\alpha_0 < \alpha_1 < \alpha_2} H^0(A_{\alpha_0, \alpha_1, \alpha_2})
$$
Deciding Connectivity

Given two points $x$ and $y$ in a set $S$;
- Decide whether they are in the same connected component of $S$.
- If yes, construct a path in $S$ joining them.

(B-Pollack-Roy, 1995) We give an algorithm to solve both problems for semi-algebraic sets restricted to a variety of dimension $k'$ in time,

$$n^{k'+1}d^{O(k^2)}.$$

(B-Pollack-Roy, 1997) We also give semi-algebraic descriptions of the connected components in time

$$n^{k+1}d^{O(k^3)}.$$
What is a Roadmap?

A roadmap of $S$, passing through a given set of points, $\mathcal{M}$, $R(S,\mathcal{M})$, is a semi-algebraic set of dimension at most one containing $\mathcal{M}$, satisfying:

1. for every semi-algebraically connected component $C$ of $S$, $C \cap R(S,\mathcal{M})$ is non-empty and semi-algebraically connected.

2. for every $x \in R$, and for every semi-algebraically connected component $C'$ of $S_x$, $C' \cap R(S,\mathcal{M})$ is non-empty, where $S_x = S \cap (X_1 = x)$. 
How to compute the roadmap?

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In case of a compact, smooth algebraic hypersurface $Z(Q)$ one can obtain the roadmap by:

**Step 1:** Follow the $X_2$-extremal points in the $X_1$ direction. Algebraically, follow parametrically the solutions of,

$$Q = \frac{\partial Q}{\partial X_3} = \cdots = \frac{\partial Q}{\partial X_k} = 0.$$
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In case of a compact, smooth algebraic hypersurface $Z(Q)$ one can obtain the roadmap by:

**Step 2:** Recurse at certain special slices corresponding to the critical values of the projection map onto the $X_1$ co-ordinate.

Algebraically, critical values are the $X_1$ co-ordinates of the real solutions of the system,

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How to compute the roadmap?

In case of a compact, smooth algebraic hypersurface $Z(Q)$ one can obtain the roadmap by:

**Step 3:** Recurse also at the $X_1$ co-ordinates of the input points.
The torus in $\mathbb{R}^3$
The torus in $\mathbb{R}^3$

Extra points in the recursive call

Critical slices

Sweep direction
The torus in $\mathbb{R}^3$

Critical slices

Extra points in the recursive call

Extra points in the input

Sweep direction
Let $Y = (Y_1, \ldots, Y_k)$ be a parametric point.
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Treating $Y$ as a parameter we compute a family of polynomials, $\mathcal{L}(Y)$, such that the signs of the polynomials in $\mathcal{L}$ determine the “type” of the connecting path $\Gamma(Y)$ connecting $Y$ to a certain distinguished point.
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Points satisfying the same sign condition on $\mathcal{L}$ lie in the same connected component of $S$. Thus, each connected component of $S$ can be described by a disjunction of sign conditions on $\mathcal{L}$.

For a fixed sign condition $\sigma$ on $\mathcal{L}$, the union of the paths $\Gamma(y)$ such that $\text{sign}\mathcal{L}(y) = \sigma$ is contractible.
Single exponential time algorithm for $b_1$

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- Compute (using the Roadmap Algorithm) the connected components of the pair-wise and triple-wise intersections of the sets in the cover.
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- $H_1(A)$ by computing the middle homology of

\[ \prod_{\alpha_0} H^0(A_{\alpha_0}) \xrightarrow{\delta} \prod_{\alpha_0 < \alpha_1} H^0(A_{\alpha_0, \alpha_1}) \xrightarrow{\delta} \prod_{\alpha_0 < \alpha_1 < \alpha_2} H^0(A_{\alpha_0, \alpha_1, \alpha_2}) \]
Algorithm for sets defined by quadratic inequalities

For any fixed $\ell > 0$, there is an algorithm which given a set of $n$ polynomials,

$$\mathcal{P} = \{P_1, \ldots, P_n\} \subset [X_1, \ldots, X_k],$$

with

$$\deg(P_i) \leq 2, 1 \leq i \leq n,$$

computes

$$b_k(S), \ldots, b_{k-\ell}(S),$$

where $S$ is the set defined by $P_1 \geq 0, \ldots, P_s \geq 0$. The complexity of the algorithm is

$$s^{\ell+2} k^{2O(\ell)}.$$
Main Idea

Consider $S$ as the intersection of the various $S_i$'s and consider the double complex arising from the generalized Mayer-Vietoris exact sequence.
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Consider $\mathcal{S}$ as the intersection of the various $S_i$’s and consider the double complex arising from the generalized Mayer-Vietoris exact sequence.

This enables us to reduce the problem of computing the top $\ell$ Betti numbers of $\mathcal{S}$, to the problem of computing certain complexes, whose homology groups are isomorphic to those of the unions of the $S_i$’s (taken at most $\ell + 2$ at a time), as well as computing certain natural homomorphisms between these complexes.
Dealing with small unions

Let $P_1, \ldots, P_s$ be homogeneous quadratic polynomials in $R[X_0, \ldots, X_k]$. We denote by

$$P = (P_1, \ldots, P_s) : R^{k+1} \rightarrow R^s.$$
Dealing with small unions

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Let

\[
\Omega = \{ \omega \in R^s \mid |\omega| = 1, \omega_i \leq 0, 1 \leq i \leq s \}.
\]

and for \( \omega \in \Omega \) let

\[
\omega P = \sum_{i=1}^{s} \omega_i P_i.
\]
Let $B \subset \Omega \times S^k$ be the set defined by,

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The map $\phi_2$ gives a homotopy equivalence between $B$ and

$$\phi_2(B) = \bigcup_{i=1}^{s} \{x \in S^k \mid P_i \leq 0\}$$
Computing the Leray Spectral Sequence of $\phi_1$

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- For any simplex $\sigma \in \Delta(\Omega)$ and $\omega \in \sigma$, $\phi_1^{-1}(\sigma)$ is homotopy equivalent to $\phi_1^{-1}(\omega)$, and both these spaces have the homotopy type of the sphere $S^k$—index($\omega P$).
Computing the Leray Spectral Sequence of $\phi_1$

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- From this observation we can compute a double complex whose associated to spectral sequence is the Leray spectral sequence of $\phi_1$. 
And finally ...

On behalf all the participants, a very big THANK YOU to the organizers, for organizing such a great conference !!