Stacks and homotopy theory

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Torsors

$S =$ “decent” scheme, ie. noetherian, locally of finite type, ...

$G =$ group-scheme defined over S, eg. Gl_n, etc.

G represents a sheaf of groups $G = \hom(\ , G)$ for the standard geometric topologies (eg. Zariski, flat, étale, Nisnevich) on $Sch|_S =$ schemes locally of fin. type/S.

G**-torsor**: sheaf X with free G-action such that $X/G \to *$ is isomorphism, $* =$ terminal sheaf.

ie. G acts freely on X, and there is sheaf epi $U \to *$ and map $\sigma : U \to X$ s.t. following dia. is a pullback:

$$\begin{array}{ccc} G \times U & \xrightarrow{\sigma_*} & X \\ {\scriptstyle pr}\downarrow & & \downarrow \\ U & \longrightarrow & * \end{array} \qquad \sigma_*(g, u) = g\sigma(u)$$

Cocycles

$\sigma(u_2) = c(u_1, u_2)\sigma(u_1)$ for uniquely determined $c(u_1, u_2) \in G$ for all sections u_1, u_2 of U.

$(u_1, u_2) \mapsto c(u_1, u_2)$ defines a cocycle c, from which the torsor X can be reassembled up to iso. from a G-equivariant coequalizer (twist one projection by c)

$$G \times U \times U \rightrightarrows G \times U \to X$$

The cocycle is a map of sheaves of groupoids $c : U_\bullet \to G$ where $U_\bullet =$ the trivial groupoid arising from the sheaf epi $U \to *$. It is also a map of simplicial sheaves

$$c : U_\bullet \to BG$$

where $U_\bullet =$ Čech resolution for $U \to *$.

Classifying objects

BC denotes the nerve of a category C.

$BC_n =$ strings $a_0 \to a_1 \to \cdots \to a_n$ with faces and degeneracies defined by composition, insertion of identities resp.

eg: $G =$group is a category with one object $*$, then n-simplices of BG are strings

$$* \xrightarrow{g_1} * \xrightarrow{g_2} \dots \xrightarrow{g_n} *$$

or elements of $G^{\times n}$.

The construction $C \mapsto BC$ is functorial and preserves the sheaf condition, so there is a simplicial sheaf BG associated to a sheaf of groups (or groupoids) G.

Homotopy theory

Simplicial homotopy corr. to isomorphism of G-torsors, so we have bijection

$$\pi(U_\bullet, BG) \cong \{\text{iso. classes of } G\text{-torsors, trivial over } U\}$$

Joyal (mid 80s): A map of simp. sheaves $X \to Y$ is a weak equiv. iff it induces a weak equiv. $X_x \to Y_x$ of simp. sets in all stalks. Cofibrations are monomorphisms. There is a model structure, with ass. htpy. category $\mathrm{Ho}(s\,\mathrm{Shv})$.

$$\begin{aligned}
[*, BG] &\cong \varinjlim_{\text{hypercovers } V \to *} \pi(V, BG) \\
&\cong \varinjlim_{\text{epi } U \to *} \pi(U_\bullet, BG) \\
&\cong \{\text{iso. classes of } G\text{-torsors}\} = H^1(S, G)
\end{aligned}$$

Examples

For the étale topology/S:

1) $[*, BGl_n] \cong \{$iso. classes of rank n vector bundles$\}$

2) $[*, BO_n] \cong \{$iso. classes of rank n sym. bil. forms$\}$

3) $[*, BPGl_n] \cong \{$iso. classes of rank n^2 Azumaya algebras$\}$.
$[*, BPGl_n] \to Br(S) =$ similarity classes.

General fact: $[*, K(A,n)] \cong H^n(S, A)$ for all abelian sheaves A.

$X =$ simplicial scheme or sheaf:

$$H^n(X, A) := [X, K(A,n)]$$

(cup products, Steenrod operations ...)

Brauer group

There is a fibre seq. $B\mathbb{G}_m \to BGl_n \to BPGl_n$

$B\mathbb{G}_m$ acts freely on BGl_n with quotient $BPGl_n$, so there is a fibre sequence

$$BGl_n \to EB\mathbb{G}_m \times_{B\mathbb{G}_m} BGl_n \to BB\mathbb{G}_m$$

$BB\mathbb{G}_m \simeq K(\mathbb{G}_m, 2)$, Borel const. $\simeq BPGl_n$, so there is an induced map

$$[*, BPGl_n] \to [*, K(\mathbb{G}_m, 2)] \text{ (n-torsion)}$$

Assembling these gives a monomorphism

$$Br(S) \to H^2(S, \mathbb{G}_m)_{tors}.$$

NB: no mention of non-abelian H^2.

Effective descent

A **stack** is a sheaf of groupoids which satisfies the effective descent condition.

Effective descent datum $\{x_\phi\}$ in G for cov. sieve $R \subset \hom(U)$

- objects $x_\phi \in G(V)$, one for each $\phi : V \to U$ in R
- isomorphisms $c_\psi : \psi^* x_\phi \to x_{\phi\psi}$ for each composable pair $W \xrightarrow{\psi} V \xrightarrow{\phi} U$

s.t. $c_1 = 1$ and following diagrams commute

$$\begin{array}{ccc} \omega^*\psi^*x_\phi & \xrightarrow{\omega^* c_\psi} & \omega^* x_{\phi\psi} \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle c_\omega} \\ (\psi\omega)^* x_\phi & \xrightarrow[c_{\psi\omega}]{} & X_{\phi\psi\omega} \end{array} \qquad W' \xrightarrow{\omega} W \xrightarrow{\psi} V \xrightarrow{\phi} U$$

Stacks

The effective descent data for R in G are members of a category $\hom(E_R, G)$ with morphisms $\{x_\phi\} \to \{y_\phi\}$ given by families $x_\phi \to y_\phi$ in $G(V)$ which are compatible with structure.

There is a functor

$$G(U) \to \hom(E_R, G)$$

defined by $x \mapsto \{\phi^* x\}$.

A sheaf of groupoids G is a **stack** if all functors $G(U) \to \hom(E_R, G)$ are equivalences of groupoids for all covering sieves R..

$\hom(E_R, G)$

$R \subset \hom(\ ,U)$ is a subfunctor, also defines a category R with objects $\phi : V \to U$ in R and comm. triangles for morphisms. There is a functor $R \to \mathrm{Shv}$ defined by sending $\phi : V \to U$ to $V = \hom(\ ,V)$.

E_R is “translation category” defined by this functor in the sheaf category.

$$E_R : \quad \bigsqcup_{W \to V \xrightarrow{\phi} U} W \rightrightarrows \bigsqcup_{V \xrightarrow{\phi} U} V$$

Effective descent datum is functor $E_R \to G$ (simplicial sheaf map $BE_R \to BG$), morphism is natural transformation (homotopy) of such functors.

Stack completion

A sheaf of groupoids G is a stack if and only if all induced maps

$$BG(U) \to \mathbf{hom}(BE_R, BG) = B\,\hom(E_R, G)$$

are weak equivalences of simplicial sets.

Stack completion:

$$St^p(G)(U) = \varinjlim_{R \subset \hom(\ ,U)} \hom(E_R, G)$$

defines a presheaf of groupoids St^pG.

The **stack completion** $St(G)$ of G is the associated sheaf for $St^p(G)$.

More homotopy theory

Joyal-Tierney (mid 80s): There is a homotopy theory for $\mathrm{Shv}(Gpd) =$ sheaves of groupoids, for which a map $f : G \to H$ is a local weak equivalence (resp. fibration) if the induced map $f : BG \to BH$ is a local weak equivalence (resp. fibration) of simplicial sheaves.

Every sheaf of groupoids G has **fibrant model** $j : G \to G^{\wedge}$ (ie. weak equiv., $BG^{\wedge}$ fibrant simplicial sheaf.

Some results

Fact: Every fibrant groupoid H is a stack.

Proof: $BE_R \to U$ is a local weak equivalence, so $\mathbf{hom}(U, BH) \to \mathbf{hom}(BE_R, BH)$ is a weak equivalence of simplicial sets.

Fact: If G is a stack and $G \to G^\wedge$ is a fibrant model, then all $G(U) \to G^\wedge(U)$ are equivalences of groupoids, ie $BG \to BG^\wedge$ is sectionwise weak equivalence.

Proof: There is isomorphism $\pi_0 BG(U) \cong [U, BG]$.

Fact: $G \to St(G)$ is a weak equivalence of sheaves of groupoids.

Proof: Descent data lift to G locally.

Corollary: The stack completion $St(G)$ and fibrant model $G^\wedge$ coincide up to natural sectionwise equivalence.

Stacks are fibrant groupoids

Fact: (old) There is a homotopy theory for $s\,\mathrm{Pre}=$ simplicial presheaves for which weak equivs. are stalkwise weak equivs. and cofibrations are monomorphisms. The ass. sheaf map $X \to \tilde{X}$ is a weak equiv. $\mathrm{Ho}(s\,\mathrm{Pre}) \simeq \mathrm{Ho}(s\,\mathrm{Shv})$.

Fact: (Hollander) There is a homotopy theory for $\mathrm{Pre}(Gpd) =$ presheaves of groupoids for which weak equivs. (resp. fibrations) are maps $G \to H$ such that $BG \to BH$ are local weak equivs. (resp. fibrations). The assoc. sheaf map $G \to \tilde{G}$ is a weak equiv. $\mathrm{Ho}(\mathrm{Pre}(Gpd)) \simeq \mathrm{Ho}(\mathrm{Shv}(Gpd))$.

Observation: Every fibrant sheaf of groupoids is a fibrant presheaf of groupoids..

Stacks are homotopy types of presheaves of groupoids

Torsors, revisited

$G =$ sheaf of groups:

$$\begin{aligned} St^p(G)(U) &\simeq \{G\text{-cocycles over } U\} \\ &\simeq \{G|_U \text{ torsors over } U\} \end{aligned}$$

Fact: $St^p(H)(U) \longrightarrow St(H)(U)$ is an equivalence for all sheaves of groupoids H, in all sections.

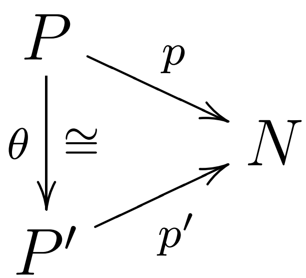
Consequence: $St(G)(U) \simeq \{G|_U \text{ torsors over } U\}$ for all U.

NB: $St(G)$ is a sheaf or presheaf of groupoids rather than a fibre functor.

Quotient stacks

$G \times N \to N$ is action by sheaf of groups on a sheaf N.

The **quotient stack** $G - \mathbf{Tors}/N$, in sections, has objects all G-equivariant maps $P \to N$ where P is a G-torsor, and has morphisms all G-equivariant diagrams



There is a groupoid of translation categories $E_G N$ arising from the G-action, and $B(E_G N) \cong EG \times_G N$.

Fact: $[*, EG \times_G N] \cong \pi_0(G - \mathbf{Tors}/N)(S)$. $St(E_G N)$ is sectionwise equivalent to $G - \mathbf{Tors}/N$.

Alternative description of torsors

$G =$ sheaf of groups: a G-torsor is a sheaf X with G-action such that the simplicial sheaf map $EG \times_G X \to *$ is a weak equivalence.

$EG \times_G X = \underrightarrow{\text{holim}}_G X$ for the diagram on groupoid G defined by G-action.

$H =$ sheaf of groupoids: an H-torsor is a functor $X : H \to \text{Shv}$ (internally defined) such that the map $\underrightarrow{\text{holim}}_H X \to *$ is a weak equiv. $[*, BH] \cong \pi_0(H - \mathbf{Tors})$.

Gerbes

A **gerbe** H is a locally connected stack (ie. $\tilde{\pi}_0(H) \cong *$).

G sheaf of groups: a G-gerbe is a gerbe locally equivalent to $G - \mathbf{Tors}$.

$\mathbf{Aut}(G)$ is two groupoid of automorphisms of G and homotopies.

Luo, J. (2004): $[*, B(\mathbf{Aut}(G)^o)] \cong \pi_0(G - \mathbf{gerbe})$

Algebraic stacks

Specialize to the étale topology for $Sch|_S$:

Laumon, Moret-Bailly: Algebraic stacks occur as stack completions of (ie. htpy types represented by) special sheaves of groupoids $X_1 \rightrightarrows X_0$, where

- X_1, X_0 are algebraic spaces (ie. étale quotients of schemes)
- the source and target maps $X_1 \to X_0$ are smooth
- the map $X_1 \to X_0 \times_S X_0$ is separated and quasi-compact.

If the source and target maps are étale, the stack is a Deligne-Mumford stack.

Example: $\mathcal{M}_{g,n} =$ moduli stack of proper smooth curves/S of genus g with n ordered points.

Stack cohomology

Perspective: $X =$ algebraic stack.

"Naive" stack cohomology $H^*(X, A)$ is defined by the étale site of a simplicial algebraic space, which is BX.

Claim:

$$H^n(X, A) \cong [BX, K(A, n)]$$

for abelian sheaves A defined on the big site $Sch|_S$.

This is a generalization of a corresponding result for simplicial schemes.

Some details

The site $et|_{BX}$ has objects all étale maps $\phi : U \to BX_n$ (algebraic spaces) and morphisms

$$\begin{array}{ccc} V & \longrightarrow & U \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \phi} \\ BX_m & \xrightarrow[\theta^*]{} & BX_n \end{array} \qquad \theta : \mathbf{n} \to \mathbf{m} \text{ in } \mathbf{\Delta}$$

Coverings are generated by étale coverings $U_i \to U$. $F =$ sheaf on $(Sch|_S)_{et}$: $F|_{BX}(\phi) = \hom(U, F)$. $F|_{BX}$ is sheaf on $et|_{BX}$ and $F \mapsto F|_{BX}$ is exact, so restriction preserves weak equivs.

Idea: show that there is a bijection $[BX, Y] \cong [*, Y|_{BX}]$.

More details

May as well assume that Y is fibrant.

Choose fibrant model $Y|_{BX} \to Z$ on $et|_{BX}$. $(*)$ Show that this map is a weak equivalence in all sections by showing that this is true for all restrictions to $et|_{BX_n}$ for all n.

1_{BX} is the simplicial sheaf on $et|_{BX}$ rep. by 1_{BX_n}, $n \geq 0$.

$$\begin{array}{ccc} V & \longrightarrow & BX_n \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle 1} \\ BX_m & \xrightarrow[\theta^*]{} & BX_n \end{array}$$

det. by θ. $1_{BX}(\psi) = \Delta^m \simeq *$, so $1_{BX} \to *$ is weak equiv.

$\hom(1_{BX}, Y|_{BX}) \to \hom(1_{BX}, Z) \simeq Z(*)$ is weak equiv. by $(*)$.
$\hom(1_{BX}, Y|_{BX}) \cong \hom(BX, Y)$. Compare π_0.