Path categories and quasi-categories

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Introduction

The identification of morphism sets in path categories of simplicial (or cubical) complexes is a central theme of concurrency theory. The path category functor, on the other hand, plays a role in the homotopy theory of quasi-categories that is roughly analogous to that of the fundamental groupoid in standard homotopy theory.

Concurrency theory is the study of models of parallel processing systems. One of the prevailing geometric forms of the theory represents systems as finite cubical complexes, called higher dimensional automata. Each r-cell of the complex K represents the simultaneous action of a group of r processors, while the holes in the complex represent constraints on the system, such as areas of memory that cannot be shared. The vertices of K are the states of the system, and the morphisms $P(K)(x,y)$ from x to y in the path category $P(K)$ are the execution paths from the state x to the state y.

There is no prevailing view of what higher homotopy invariants should mean in concurrency theory, or even what these invariants should be. The path category functor is not an artifact of standard homotopy theory, but it is a central feature of the theory of quasi-categories. The homotopy theory of quasi-categories is constructed abstractly within the category of simplicial sets by methods that originated in homotopy localization theory, and its weak equivalences are not described by homotopy groups.

Is the homotopy theory of quasi-categories the right venue for the study of higher homotopy phenomena in concurrency theory? Well maybe, but a definitive answer is not presented here.

It is a fundamental aspect of this flavour of homotopy theory that if a map $X \to Y$ of simplicial sets is a quasi-category weak equivalence (a "categorical weak equivalence" below), then the induced functor $P(X) \to P(Y)$ is an equivalence of categories — see Lemma 28. This phenomenon may be a bit strong for computational purposes, given that recent complexity reduction techniques for concurrency algorithms focus on producing subcomplexes $L \subset K$ of a sim-

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plicial complex K which induce fully faithful imbeddings $P(L) \to P(K)$ of the associated path categories.

It nevertheless appears that the homotopy theory of quasi-categories is a good first approximation of a theory that is suitable for the homotopy theoretic analysis of the computing models that are represented by higher dimensional automata.

Further, recent work of Nicholas Meadows [7] shows that the theory of quasi-categories can be extended to a homotopy theory for simplicial presheaves whose weak equivalences are those maps which are stalkwise quasi-category weak equivalences. Meadows' theory starts to give a local to global picture of systems that are infinite in the practical sense that they cannot be studied by ordinary algorithms.

This paper is esssentially expository: it is an introduction to the "ordinary" homotopy theory of quasi-categories. The exposition presented here has evolved from research notes that were written at a time when complete descriptions of Joyal's quasi-category model structure for simplicial sets were not publicly available. The overall line of argument is based on a method which first appeared Cisinski's thesis [1], [2], and is now standard. This presentation is rapid, and is at times aggressively combinatorial. Some readers may prefer other descriptions of the theory, such as one finds in [5] or [6].

From a calculational point of view, the most interesting result of this paper might be Proposition 38, which characterizes weak equivalences between quasi-categories as maps which induce equivalences on a naturally defined system of groupoids. These groupoids are effectively the higher homotopy "groups" for quasi-category homotopy theory.

The collection of ideas appearing in the argument for Proposition 38 also applies within the ordinary homotopy theory of simplicial sets. Corollary 39 says that there is a system of naturally defined groupoids for Kan complexes, such that a map of Kan complexes is a standard weak equivalence if and only if it induces equivalences of this list of associated groupoids. This criterion is base point free, but may not be all that new — it can be proved directly with standard techniques of simplicial homotopy theory.

Contents

1 The path category functor

Suppose that X is a simplicial set. The *path category* $P(X)$ of X is the category freely that is generated by the graph defined by the vertices and edges of X, subject to the relations

$$d_1(\sigma) = d_0(\sigma) \cdot d_2(\sigma),$$

one for each 2-simplex σ of X, and $s_0(x) = 1_x$ for each vertex x of X.

The category $P(X)$ has morphism sets $P(X)(x, y)$ defined by equivalence classes of strings of 1-simplices

$$\alpha : \ x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} x_n = y,$$

where the equivalence relation is generated by relations of the form $\alpha \sim \alpha'$ occurring in the presence of a 2-simplex σ of X with boundary

$$\begin{tikzcd} x_i \arrow[rr, "\beta"] \arrow[dr, "\alpha_{i+1}"'] & & x_{i+2} \\ & x_{i+1} \arrow[ur, "\alpha_{i+2}"'] & \end{tikzcd}$$

and where α' is the string

$$\alpha' : \ x_0 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_i} x_i \xrightarrow{\beta} x_{i+2} \xrightarrow{\alpha_{i+3}} \dots \xrightarrow{\alpha_n} x_n.$$

Composition in $P(X)$ is defined by concatenation of representing strings.

The resulting functor $X \mapsto P(X)$ is left adjoint to the nerve functor

$$B : \mathbf{cat} \to s\mathbf{Set},$$

essentially since the nerve functor takes values in 2-coskeleta. The inclusion $\mathrm{sk}_2(X) \subset X$ induces an isomorphism of categories

$$P(\mathrm{sk}_2(X)) \cong P(X).$$

Given a simplicial set map $f : X \to BC$, the adjoint functor $f_* : P(X) \to C$ is the map $f : X_0 \to \mathrm{Ob}(C)$ in degree 0, and that takes a 1-simplex $\alpha : d_1(\alpha) \to d_0(\alpha)$ to the morphism $f(\alpha) : f(d_1(\alpha)) \to f(d_0(\alpha))$ of C. In particular, the canonical functor $\epsilon : P(BC) \to C$ is the identity on objects, and takes a 1-simplex $\alpha : x \to y$ of BC to the corresponding morphism of C.

Lemma 1. *The canonical functor*

$$\epsilon : P(BC) \to C$$

is an isomorphism for each small category C.

Proof. There is a functor $s : C \to P(BC)$, which is the identity on objects, and takes a morphism $\alpha : x \to y$ to the morphism that is represented by the 1-simplex α of BC. The composition law is preserved, because the 2-simplices of BC are composition laws. The morphism of $P(BC)$ that is represented by a string of 1-simplices

$$x = x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} x_n = y$$

in BC is also represented by the composite $\alpha_n \cdots \alpha_1$, and it follows that the functor s is full. But $\epsilon \cdot s = 1_C$, so that s is faithful as well as full, and is therefore an isomorphism of categories. □

Lemma 2. *The canonical functor*

$$P(X \times Y) \to P(X) \times P(Y)$$

is an isomorphism for all simplicial sets X and Y.

Proof. If C and D are small categories, then there is a diagram

$$\begin{tikzcd}
P(B(C \times D)) \arrow[r] \arrow[d, "\epsilon"', "\cong"] & P(BC) \times P(BD) \arrow[d, "\cong"', "\epsilon \times \epsilon"] \\
C \times D \arrow[r, "1"'] & C \times D
\end{tikzcd}$$

by Lemma 1, so the claim holds for $X = BC$ and $Y = BD$. In general $X \times Y$ is a colimit of products

$$\Delta^n \times \Delta^m = B(\mathbf{n}) \times B(\mathbf{m}),$$

and so the comparison map is a colimit of the isomorphisms

$$P(\Delta^n \times \Delta^m) \xrightarrow{\cong} P(\Delta^n) \times P(\Delta^m).$$

□

Corollary 3. *1) Every simplicial homotopy $h : X \times \Delta^1 \to Y$ between maps $f, g : X \to Y$ induces a natural transformation $h_* : P(X) \times \mathbf{1} \to P(Y)$ between the corresponding functors $f_*, g_* : P(X) \to P(Y)$.*

2) Every simplicial homotopy equivalence $X \to Y$ induces a homotopy equivalence of categories $P(X) \to P(Y)$.

3) Every trivial Kan fibration $\pi : X \to Y$ induces a strong deformation retraction $\pi_* : P(X) \to P(Y)$ of $P(X)$ onto $P(Y)$. The functor π_* is a homotopy equivalence of categories.

A functor $f : C \to D$ between small categories is a *homotopy equivalence* if the induced map $BC \to BD$ is a homotopy equivalence. This means that there is a functor $g : D \to C$ and natural transformations $C \times \mathbf{1} \to C$ and $D \times \mathbf{1} \to D$ between $g \cdot f$ and 1_C and between $f \cdot g$ and 1_D. Note that the direction of these natural transformations is not specified.

Proof of Corollary 3. Statement 1) is a consequence of Lemma 1 and Lemma 2.

For statement 2), we are assuming the existence of a simplicial set map $g : Y \to X$, together with simplicial homotopies $X \times \Delta^1 \to X$ between gf and 1_X and $Y \times \Delta^1 \to Y$ between fg and 1_Y. The claim follows from statement 1).

To prove statement 3), it is a standard observation that π has a section $\sigma : Y \to X$ along with a simplicial homotopy $h : X \times \Delta^1 \to X$ between $\sigma\pi$ and 1_X, so that Y is a strong deformation retract of X. More explicitly, h is constructed by finding the lifting in the diagram

$$\begin{tikzcd}
X \times \partial\Delta^1 \arrow[rr, "{(1_X, \sigma\pi)}"] \arrow[d] & & X \arrow[d, "\pi"] \\
X \times \Delta^1 \arrow[r, "\pi \times 1"'] \arrow[urr, dotted, "h"] & Y \times \Delta^1 \arrow[r, "pr"'] & Y
\end{tikzcd}$$

where the projection *pr* is a constant homotopy. If $x \in X_0$, then the composite

$$\Delta^1 \xrightarrow{(x,1)} X \times \Delta^1 \xrightarrow{h} X$$

is a 1-simplex of the fibre $F_{\pi(x)}$ over $\pi(x)$. This fibre is a Kan complex, so that the path is invertible in $P(F_{\pi(x)})$ and hence in $P(X)$. It follows that the induced natural transformation

$$h_* : P(X) \times \mathbf{1} \to P(X)$$

is a natural isomorphism. □

The set $\pi(X, Y)$ of naive homotopy classes between simplicial sets X and Y is the set of path components of the function space $\mathbf{hom}(X, Y)$. Generally the set of path components $\pi_0(X)$ of a simplicial set X coincides with the set

$$\pi_0 P(X) = \pi_0(B(P(X)))$$

of path components of the path category of X. We shall be interested in something stronger, namely the set

$$\tau_0 P(X)$$

of isomorphism classes of $P(X)$.

Say that maps $f, g : X \to Y$ are *strongly homotopic* if there is an isomorphism $f \xrightarrow{\cong} g$ in the path category $P(\mathbf{hom}(X, Y))$, and write

$$\tau_0(X, Y) = \tau_0 P(\mathbf{hom}(X, Y)).$$

Suppose that X is a Kan complex. Then a morphism of $P(X)$ that is represented by a string of 1-simplices

$$x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} x_n,$$

is also represented by a 1-simplex $x_0 \to x_n$. In effect, the string

$$x_0 \xrightarrow{\alpha_1} x_1 \xrightarrow{\alpha_2} x_2$$

defines a simplicial set map $\Lambda^2_1 \to X$ that extends to a 2-simplex $\sigma : \Delta^2 \to X$, and so the morphism represented by the string (α_1, α_2) is also represented by the 1-simplex $d_1(\sigma) : x_0 \to x_2$.

Lemma 4. *Suppose that X is a Kan complex. Then $P(X)$ is a groupoid.*

Proof. Every morphism of $P(X)$ is represented by a 1-simplex $\alpha : x \to y$ of X. The solutions of the lifting problems

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "(\ {,} s_0(x){,}\alpha)"] \arrow[d] & X \\ \Delta^2 \arrow[ur, dotted] & \end{tikzcd}$$

and

$$\begin{tikzcd} \Lambda^2_2 \arrow[r, "(\alpha{,}s_0(y){,} \ )"] \arrow[d] & X \\ \Delta^2 \arrow[ur, dotted] & \end{tikzcd}$$

imply that there are 1-simplices $g : y \to x$ and $h : y \to x$ (respectively) such that $g \cdot \alpha = 1_x$ and $\alpha \cdot h = 1_y$ in $P(X)$. But then

$$g = g \cdot \alpha \cdot h = h$$

so that α has an inverse in $P(X)$. $\square$

Joyal shows in [4] (see also Corollary 17 below) that if X is a quasi-category (see Section 2), then the path category $P(X)$ is a groupoid if and only if X is a Kan complex.

Remark 5. Lemmas 1 and 4 together imply that the functor $X \mapsto P(X)$ does not preserve standard weak equivalences. In other words, it is not the case that a weak equivalence $f : X \to Y$ necessarily induces a weak equivalence $BP(X) \to BP(Y)$. Otherwise, the natural weak equivalence $X \simeq B(\mathbf{\Delta}/X)$ and any fibrant model $j : X \to Z$ (Z a Kan complex) would together give weak equivalences

$$X \simeq B(\mathbf{\Delta}/X) \simeq BP(X) \simeq BP(Z),$$

while $BP(Z)$ has only trivial homotopy groups in degrees above 1.

Lemma 6. *Suppose that $i : \Lambda^n_k \subset \Delta^n$ is the standard inclusion, where $n \geq 2$ and $k \neq 0, n$. Then the induced functor*

$$i_* : P(\Lambda^n_k) \to P(\Delta^n)$$

is an isomorphism.

Proof. Recall that the canonical map

$$P(\Delta^n) = P(B(\mathbf{n})) \xrightarrow{\epsilon} \mathbf{n}$$

is an isomorphism. A functor $\alpha : \mathbf{n} \to C$ can be identified with the string of arrows

$$\alpha(0) \xrightarrow{\alpha(0\leq 1)} \alpha(1) \xrightarrow{\alpha(1\leq 2)} \dots \xrightarrow{\alpha(n-1\leq n)} \alpha(n).$$

It follows that the 1-simplices $i \leq i+1 : \Delta^1 \to \Delta^n$ together define a simplicial set inclusion

$$\gamma_n : \mathrm{Path}_n = \Delta^1 * \dots * \Delta^1 \subset \Delta^n$$

defined on the join Path_n of n copies of Δ^1 (end to end), which induces an isomorphism

$$\gamma_{n*} : P(\mathrm{Path}_n) \xrightarrow{\cong} P(\Delta^n) \cong \mathbf{n}.$$

All 1-simplices $i \leq i+1$ of Δ^n are members of Λ^n_k since $k \neq 0, 1$: in effect, $i \leq i+1$ is a member of $d^n(\Delta^{n-1})$ if $i \leq n-1$, and $i \leq i+1$ is in $d^0(\Delta^{n-1})$ if $i \geq 1$. It follows that there is a commutative diagram

$$\begin{tikzcd} & \Lambda^n_k \arrow[d, "i"] \\ \mathrm{Path}_n \arrow[ur] \arrow[r, "\gamma_n"'] & \Delta^n \end{tikzcd}$$

and hence a commutative diagram

$$\begin{tikzcd} & P(\Lambda^n_k) \arrow[d, "i_*"] \\ P(\mathrm{Path}_n) \arrow[ur, "\hat{\gamma}"] \arrow[r, "\cong", "\gamma_{n*}"'] & P(\Delta^n) \end{tikzcd}$$

Every ordinal number map $\theta : \mathbf{m} \to \mathbf{n}$ determines a commutative diagram

$$\begin{tikzcd} P(\mathrm{Path}_m) \arrow[r, "\tilde{\theta}"] \arrow[d, "\gamma_{m*}"', "\cong"] & P(\mathrm{Path}_n) \arrow[d, "\cong"', "\gamma_{n*}"] \\ P(\Delta^m) \arrow[r, "\theta_*"'] & P(\Delta^n) \end{tikzcd}$$

where $\tilde{\theta}$ sends the 1-simplex $i \leq i+1$ to the composite morphism

$$\theta(i) \leq \theta(i)+1 \leq \theta(i)+2 \leq \dots \leq \theta(i+1).$$

It follows that every 1-simplex $i \leq j$ of Λ^n_k, which is in some face $d^r(\Delta^{n-1})$, is in the image of the functor

$$\hat{\gamma} : P(\mathrm{Path}_n) \to P(\Lambda^n_k).$$

The functor $\hat{\gamma}$ is therefore surjective on morphisms, and is an isomorphism. □

2 Quasi-categories

The class of *inner anodyne extensions* in simplicial sets is the saturation of the set of morphisms

$$i : \Lambda^n_k \subset \Delta^n,\ k \neq 0, n.$$

A *quasi-category* is a simplicial set X such that the map $X \to *$ has the right lifting property with respect to all inner anodyne extensions. A map $p : X \to Y$ that has the right lifting property with respect to all inner anodyne extensions is called an *inner fibration*.

Example: Suppose that C is a small category. Then BC is a quasi-category, by Lemma 6.

Here's an observation:

Lemma 7. *Suppose that $i : A \to B$ is an inner anodyne extension. Then the map $i : A_0 \to B_0$ on vertices is a bijection.*

Proof. The class of monomorphisms $j : E \to F$ that are bijections on vertices is saturated and includes all inclusions $\Lambda^n_k \subset \Delta^n$, $0 < k < n$.

To see this last claim, observe that every vertex i of Δ^n is in the face $d^j : \Delta^{n-1} \to \Delta^n$ if $j \neq i$. Thus, if $n \geq 2$ then the vertex i is in at least two faces of Δ^n. ∎

The following result is more serious:

Lemma 8. *Suppose that S is a proper subset of the $(n-1)$-simplices d^i in Δ^n that contains d^0 and d^n, and let $\langle S \rangle$ be the subcomplex of Δ^n that is generated by the simplices in S. Then the inclusion $\langle S \rangle \subset \Delta^n$ is inner anodyne.*

Proof. The proof is by decreasing induction on the cardinality of S and increasing induction on the dimension n. Observe that $2 \leq |S| \leq n$, and $\langle S \rangle = \Lambda^n_k$ for some k if $|S| = n$.

Suppose that S' is obtained from S by adding a simplex d^k, where $0 < k < n$. Then the intersection

$$\langle d^k \rangle \cap \langle S' \rangle$$

is the subcomplex of $\Delta^{n-1} \cong \langle d^k \rangle$ that is generated by the set S'' of simplices $d^k d^i$ and $d^k d^{j-1}$, where the simplices d^i, d^j are the members of S' with $i < k$ and $j > k$, respectively. In particular, the bottom and top faces $d^k d^0$ and $d^k d^{n-1}$ of $\langle d^k \rangle$ are in S'', and $|S''| = |S'| - 1 < n - 1$.

There is a pushout diagram

$$\begin{tikzcd} \langle S'' \rangle \arrow[r] \arrow[d, "i"'] & \langle S' \rangle \arrow[d] \\ \Delta^{n-1} \arrow[r, "d^k"'] & \langle S \rangle \end{tikzcd}$$

The inclusion i is inner anodyne by induction on dimension, so that the inclusion $\langle S' \rangle \subset \langle S \rangle$ is inner anodyne. Thus, since $\langle S \rangle \subset \Delta^n$ is inner anodyne, the inclusion $\langle S' \rangle \subset \Delta^n$ is inner anodyne as well. $\square$

The following result is proved in Section 4 (Theorem 45) below. The proof is somewhat delicate, and makes heavy use of Lemma 8.

Theorem 9. *Suppose that $0 < k < n$. Then the inclusion*

$$(\Lambda^n_k \times \Delta^m) \cup (\Delta^n \times \partial\Delta^m) \subset \Delta^n \times \Delta^m$$

is inner anodyne.

Remark 10. The inclusion $\Delta^0 \to \Delta^1$ of a vertex induces a map

$$(\Delta^0 \times \Delta^1) \cup (\Delta^1 \times \partial\Delta^1) \subset \Delta^1 \times \Delta^1,$$

which is not inner anodyne. This observation is a consequence of Lemma 6, since the induced map of path categories is not an isomorphism.

Corollary 11. *1) Suppose that $A \subset B$ is an inclusion of simplicial sets, and that X is a quasi-category. Then the map*

$$i^* : \mathbf{hom}(B, X) \to \mathbf{hom}(A, X)$$

is an inner fibration. If the map $i : A \to B$ is an inner anodyne extension then the map i^* is a trivial Kan fibration.

2) Suppose that X is a quasi-category and that K is a simplicial set. Then the function complex $\mathbf{hom}(K, X)$ is a quasi-category.

Suppose again that X is a quasi-category, and suppose that $\alpha, \beta : \Delta^1 \to X$ are 1-simplices $x \to y$. A right homotopy $\alpha \Rightarrow_R \beta$ is a 2-simplex $\sigma : \Delta^2 \to X$ with boundary

$$\begin{tikzcd} & y \arrow[dr, "s_0(y)"] & \\ x \arrow[ur, "\alpha"] \arrow[rr, "\beta"'] & & y \end{tikzcd}$$

and a left homotopy $\beta \Rightarrow_L \alpha$ is a 2-simplex of X with boundary

$$\begin{tikzcd} & x \arrow[dr, "\alpha"] & \\ x \arrow[ur, "s_0(x)"] \arrow[rr, "\beta"'] & & y \end{tikzcd}$$

Then the following are equivalent (by suitable choices of 3-simplices):

a) there is a right homotopy $\alpha \Rightarrow_R \beta$,

b) there is a right homotopy $\beta \Rightarrow_R \alpha$,

c) there is a left homotopy $\beta \Rightarrow_L \alpha$,

d) there is a left homotopy $\alpha \Rightarrow_L \beta$.

If any one of these conditions holds, say that α is homotopic to β, and write $\alpha \simeq \beta$.

Write $\mathrm{ho}(X)$ for the category whose objects are the vertices of X, whose morphisms $[\alpha] : x \to y$ are the homotopy classes of paths $\alpha : x \to y$ in X, and with composition law

$$\mathrm{ho}(X)(x, y) \times \mathrm{ho}(X)(y, z) \to \mathrm{ho}(X)(x, z)$$

defined for classes $[\alpha] : x \to y$ and $[\beta] : y \to z$ by

$$[\beta] \cdot [\alpha] = [d_1(\sigma)]$$

where $\sigma : \Delta^2 \to X$ is a choice of extension, as in the diagram

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "{(\beta,\ ,\alpha)}"] \arrow[d, "i"'] & X \\ \Delta^2 \arrow[ur, dotted, "\sigma"'] & \end{tikzcd}$$

One must show that the class $[d_1(\sigma)]$ is independent of the choices that are made.

Lemma 12. *There is an isomorphism of categories*

$$P(X) \cong \mathrm{ho}(X)$$

for all quasi-categories X.

Proof. The functor $P(X) \to \mathrm{ho}(X)$ is induced by the assignment $\alpha \mapsto [\alpha]$ for paths α, while all members of a homotopy class represent the same morphism in $P(X)$. Thus, there is a functor $\mathrm{ho}(X) \to P(X)$, and the two functors are inverse to each other. $\square$

It follows (choose some 3-simplices) that if $\alpha : \Delta^1 \to X$ is invertible in $P(X)$, where X is a quasi-category, then there is a path $\beta : \Delta^1 \to X$ together with 2-simplices $\sigma, \sigma' : \Delta^2 \to X$ having respective boundaries

$$\begin{tikzcd} & y \arrow[dr, "\beta"] & \\ x \arrow[ur, "\alpha"] \arrow[rr, "s_0(x)"'] & & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "\alpha"] & \\ y \arrow[ur, "\beta"] \arrow[rr, "s_0(y)"'] & & y \end{tikzcd}$$

Under these circumstances, say that the 1-simplex α is a *quasi-isomorphism* of X, as is β.

In the presence of such 2-simplices for a path $\alpha : \Delta^1 \to X$ in an arbitrary simplicial set X, the corresponding morphism α is invertible in the path category $P(X)$.

Say that a simplicial set map $p : X \to Y$ is a *right fibration* if p has the right lifting property with respect to all inclusions $\Lambda^n_k \subset \Delta^n$ with $k > 0$. This definition is consistent with [6].

Lemma 13. *Suppose that $p : X \to Y$ is a right fibration, and that X and Y are quasi-categories. Suppose that $\alpha : \Delta^1 \to X$ is a 1-simplex of X such that $p(\alpha)$ is a quasi-isomorphism. Then α is a quasi-isomorphism.*

We also say that the map p of Lemma 13 *creates* quasi-isomorphisms.

Proof. The simplex $p(\alpha)$ has a right inverse, so there is a 2 simplex $\sigma : \Delta^2 \to Y$ with boundary $\partial\sigma = (p(\alpha), p(y), \gamma)$. The lifting exists in the diagram

$$\begin{tikzcd}
\Lambda^2_2 \arrow[r, "{(\alpha,y,\ )}"] \arrow[d] & X \arrow[d, "p"] \\
\Delta^2 \arrow[r, "\sigma"'] \arrow[ur, dotted, "\theta"] & Y
\end{tikzcd}$$

so that α has a right inverse $\zeta = d_2\theta$ in $P(X)$. Similarly ζ has a right inverse ω in $P(X)$. Thus,

$$\alpha = \alpha \cdot \zeta \cdot \omega = \omega$$

in $P(X)$, so that α is a quasi-isomorphism. $\square$

Write $\Delta^m * \Delta^n \cong \Delta^{m+n+1}$ for the (poset) join of the simplices Δ^n and Δ^m. The join $X * Y$ of the simplicial sets X and Y is defined by the colimit formula

$$X * Y = \varinjlim_{\Delta^m \to X,\ \Delta^n \to Y} \Delta^m * \Delta^n.$$

where the colimit is computed over the product $\mathbf{\Delta}/X \times \mathbf{\Delta}/Y$ of the respective simplex categories. It is relatively easy to show that the maps

$$\begin{gathered}
(\Lambda^m_k * \Delta^n) \cup (\Delta^m * \partial\Delta^n) \to \Delta^m * \Delta^n \\
(\partial\Delta^m * \Delta^n) \cup (\Delta^m * \partial^n_k) \to \Delta^m * \Delta^n \\
(\partial\Delta^m * \Delta^n) \cup (\Delta^m * \partial\Delta^n) \to \Delta^m * \Delta^n
\end{gathered}$$

are isomorphic to the maps $\Lambda^{m+n+1}_k \subset \Delta^{m+n+1}$, $\Lambda^{m+n+1}_{m+k+1} \subset \Delta^{m+n+1}$, and $\partial\Delta^{m+n+1} \subset \Delta^{m+n+1}$, respectively.

Suppose that $\mathbf{hom}_j(X, Y)$ is the simplicial set with r-simplices given by the maps $\Delta^r * X \to Y$. If $p : X \to Y$ is an inner fibration then the induced map

$$(i^*, p_*) : \mathbf{hom}_j(\Delta^n, X) \to \mathbf{hom}_j(\partial\Delta^n, X) \times_{\mathbf{hom}_j(\partial\Delta^n, Y)} \mathbf{hom}_j(\Delta^n, Y)$$

is a right fibration provided that $n \geq 0$. It follows (by setting $Y = *$) that all maps

$$\mathbf{hom}_j(\Delta^n, X) \to \mathbf{hom}_j(\partial\Delta^n, X)$$

are right fibrations if X is a quasi-category.

Proposition 14. *Suppose given a commutative solid arrow diagram*

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\alpha"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where $n \geq 2$, X and Y are quasi-categories, p is an inner fibration, and the map α takes the 1-simplex $0 \to 1$ to a quasi-isomorphism of X. Then the dotted arrow lifting exists, making the diagram commute.

Proof. The lifting problem

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\alpha"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

is isomorphic to a lifting problem

$$\begin{tikzcd} \Lambda^1_0 \arrow[r, "x"] \arrow[d] & \mathbf{hom}_j(\Delta^{n-2}, X) \arrow[d, "{(i^*,p_*)}"] \\ \Delta^1 \arrow[r, "\alpha_*"'] \arrow[ur, dotted] & \mathbf{hom}_j(\partial\Delta^{n-2}, X) \times_{\mathbf{hom}_j(\partial\Delta^{n-2},Y)} \mathbf{hom}_j(\Delta^{n-2}, Y) \end{tikzcd}$$

where $i : \partial\Delta^{n-2} \subset \Delta^n$ The map (i^*, p_*) is a right fibration, as is the projection map

$$\mathbf{hom}_j(\partial\Delta^{n-2}, X) \times_{\mathbf{hom}_j(\partial\Delta^{n-2},Y)} \mathbf{hom}_j(\Delta^{n-2}, Y) \to \mathbf{hom}_j(\partial\Delta^{n-2}, X).$$

The inclusion of any vertex of $\partial\Delta^{n-2}$ induces a right fibration

$$\mathbf{hom}_j(\partial\Delta^{n-2}, X) \to X.$$

It follows from Lemma 13 that the 1-simplex α_* is a quasi-isomorphism.

Let β be an inverse for α_*. The lifting problem

$$\begin{tikzcd} \Lambda^1_1 \arrow[r, "x"] \arrow[d] & \mathbf{hom}_j(\Delta^{n-2}, X) \arrow[d, "{(i^*,p_*)}"] \\ \Delta^1 \arrow[r, "\beta"'] \arrow[ur, dotted, "\theta"] & \mathbf{hom}_j(\partial\Delta^{n-2}, X) \times_{\mathbf{hom}_j(\partial\Delta^{n-2},Y)} \mathbf{hom}_j(\Delta^{n-2}, Y) \end{tikzcd}$$

has a solution since the map (i^*, p_*) is a right fibration. It follows from Lemma 13 that θ is a quasi-isomorphism of $\mathbf{hom}_j(\Delta^{n-2}, X)$. Let γ be an inverse of θ. Then $i^*(\gamma) \sim \alpha_*$ so that α lifts along i^*. $\square$

Here's the dual statement:

Corollary 15. *Suppose given a commutative solid arrow diagram*

$$\begin{tikzcd} \Lambda^n_n \arrow[r, "\alpha"] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

where $n \geq 2$, X and Y are quasi-isomorphisms, p is an inner fibration, and the map α takes the 1-simplex $n-1 \to n$ to a quasi-isomorphism of X. Then the dotted arrow lifting exists, making the diagram commute.

Corollary 16. *Suppose that X is a quasi-category and suppose that $n \geq 2$. Then the liftings exist in the diagrams*

$$\begin{tikzcd} \Lambda^n_0 \arrow[r, "\alpha"] \arrow[d] & X \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n_n \arrow[r, "\beta"] \arrow[d] & X \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

provided that α takes the simplex $0 \to 1$ to a quasi-isomorphism of X, respectively that β takes the simplex $n-1 \to n$ to a quasi-isomorphism of X.

Corollary 17. *Suppose that X is a quasi-category. Then we have the following:*

1) *If $P(X)$ is a groupoid, then X is a Kan complex. Thus, a quasi-category X is a Kan complex if and only if $P(X)$ is a groupoid.*

2) *Let $J(X) \subset X$ be the subobject consisting of those simplices $\sigma : \Delta^n \to X$ such that all composites*
$$\Delta^1 \to \Delta^n \xrightarrow{\sigma} X$$
represent isomorphisms of $P(X)$. Then $J(X)$ is a Kan complex, and it is the maximal Kan subcomplex of X.

The subcomplex $J(X)$ is often called the *core* of the quasi-category X.

Corollary 18. *Suppose that $p : X \to Y$ is an inner fibration, where X and Y are Kan complexes. Suppose also that p has the path lifting property. Then p is a Kan fibration.*

Proof. The lifting problems

$$\begin{tikzcd} \Lambda^n_0 \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} \qquad \begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

have solutions for $n = 1$ by assumption, and have solutions for $n > 1$ by the results above, since every 1-simplex of a Kan complex is a quasi-isomorphism. □

Suppose that X is a simplicial set, and write $\mathbf{hom}_I(\Delta^1, X)$ for the simplicial set whose n-simplices are the maps $\alpha : \Delta^1 \times \Delta^n \to X$ such that all vertices x of Δ^n determine composites

$$\Delta^1 \xrightarrow{(1,x)} \Delta^1 \times \Delta^n \xrightarrow{\alpha} X,$$

which represent isomorphisms of the path category $P(X)$. Write $I = B\pi(\Delta^1))$, and observe that $\pi(\Delta^1)$ is the free groupoid on the ordinal number $\mathbf{1}$, and is therefore the groupoid that is freely generated by one non-trivial arrow $\eta : 0 \to 1$. The path

$$\Delta^1 \xrightarrow{\eta} I$$

induces a map

$$\mathbf{hom}(I, X) \xrightarrow{\eta^*} \mathbf{hom}_I(\Delta^1, X).$$

Lemma 19. *Suppose that X is a quasi-category and let $n > 0$. Then the inner fibration*

$$i^* : \mathbf{hom}(\Delta^n, X) \to \mathbf{hom}(\partial\Delta^n, X)$$

induced by the inclusion $i : \partial\Delta^n \subset \Delta^n$ creates quasi-isomorphisms.

Remark 20. Lemma 19 implies that the n-simplices of $\mathbf{hom}_I(\Delta^1, X)$ for a quasi-category X are exactly the quasi isomorphisms $\Delta^n \times \Delta^1 \to X$ of the quasi-category $\mathbf{hom}(\Delta^n, X)$, provided that X is a quasi-category.

Proof of Lemma 19. Suppose that $\alpha : \Delta^n \times \Delta^1 \to X$ is a morphism such that the composite

$$\alpha_* : \partial\Delta^n \times \Delta^1 \xrightarrow{i \times 1} \Delta^n \times \Delta^1 \xrightarrow{\alpha} X$$

is a quasi-isomorphism of $\mathbf{hom}(\partial\Delta^n, X)$. Then there is a 1-simplex $\beta : \partial\Delta^n \times \Delta^1 \to X$, and a 2-simplex $\sigma : \partial\Delta^n \times \Delta^2 \to X$ of $\mathbf{hom}(\partial\Delta^n, X)$ such that

$$\partial(\sigma) = (d_0\sigma, d_1\sigma, d_2\sigma) = (\beta, s_0(d_1\alpha_*), \alpha_*).$$

We therefore have an induced map

$$(\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_0) \xrightarrow{(\sigma,(\ ,s_0(d_1\alpha),\alpha))} X$$

and I claim that it suffices to show that this map extends to a morphism $\theta : \Delta^n \times \Delta^2 \to X$.

If so, then $\beta' = d_1\sigma$ is a left inverse for α in $P(\mathbf{hom}(\Delta^n, X)$, and so every simplex α such that $i^*(\alpha)$ is a quasi-isomorphism has a left inverse in the path category, and is therefore monic. But then β' is also monic in the path category, so that

$$\beta' \cdot \alpha \cdot \beta' = \beta'$$

forces $\alpha \cdot \beta' = 1$, so that α is a quasi-isomorphism.

We must therefore solve the extension problem

$$\begin{tikzcd} (\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_0) \arrow[r, "{(\sigma,(\ ,s_0(d_1\alpha),\alpha))}"] \arrow[d] & X \\ \Delta^n \times \Delta^2 \arrow[ur, dotted] & \end{tikzcd}$$

We do this by using the ordering on the set of non-degenerate simplices of $\Delta^n \times \Delta^2$ of dimension $n+2$ that is developed in connection with the proof of Theorem 45 below.

A non-degenerate $(n+2)$-simplex

$$(0,0) \to \cdots \to (n,2)$$

is a path in the poset $\mathbf{n} \times \mathbf{2}$ with first segment either $(0,0) \to (1,0)$ or $(0,0) \to (0,1)$. Let S_0 be the set of non-degenerate $(n+2)$-simplices starting with $(0,0) \to (1,0)$ and let S' be those non-degenerate $(n+2)$-simplices starting with $(0,0) \to (0,1)$. The path

$$P: \quad \begin{tikzcd} & (1,2) \arrow[r] & \dots \arrow[r] & (n,2) \\ & (1,1) \arrow[u] & & \\ (0,0) \arrow[r] & (1,0) \arrow[u] & & \end{tikzcd}$$

is the minimal simplex of S_0, and the path

$$Q_n: \quad \begin{tikzcd} & & (n,2) \\ (0,1) \arrow[r] & \dots \arrow[r] & (n,1) \arrow[u] \\ (0,0) \arrow[u] & & \end{tikzcd}$$

is the maximal simplex of S'.

If T is a set of non-degenerate $(n+2)$-simplices, write

$$(\Delta^n \times \Delta^2)^{(T)}$$

for the subcomplex of $\Delta^n \times \Delta^2$ that is generated by the subcomplex

$$(\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_0)$$

and the members of T.

All members P' of S_0 have $d_0 P'$ and $d_{n+2}P'$ in the subcomplex

$$(\Delta^n \times \Delta^2)^{(\emptyset)} = (\partial\Delta^n \times \Delta^2) \cup (\Delta^n \times \Lambda^2_0),$$

so the inclusion

$$(\Delta^n \times \Delta^2)^{(\emptyset)} \subset (\Delta^n \times \Delta^2)^{(S_0)}$$

is inner anodyne by Lemma 8.

Suppose that Q_i is the $(n+2)$-simplex

$$\begin{tikzcd} & & (i,2) \arrow[r] & \dots \arrow[r] & (n,2) \\ (0,1) \arrow[r] & \dots \arrow[r] & (i,1) \arrow[u] & & \\ (0,0) \arrow[u] & & & & \end{tikzcd}$$

and write

$$S'_i = S \cup \{Q_0, \dots, Q_i\}.$$

Suppose first that $i < n$.

The face $d_0 Q_i$ has faces $d_0 d_0 Q_i$ and $d_{n+1} d_0 Q_i$ in the the image of the boundary subcomplex $\partial(\Delta^n \times \Delta^1)$ under the map $1 \times d^0 : \Delta^n \times \Delta^1 \to \Delta^n \times \Delta^2$, and the intersection of the boundary of $d_0 Q_i$ with $(\Delta^n \times \Delta^2)^{(S'_{i-1})}$ is missing the face $d_{i+1} d_0 Q_i$.

If $i < n$, then $d_{i+2} Q_i$ is an interior simplex and it is only a face of Q_i and Q_{i+1}, so that $d_{i+2} Q_i$ is not in the subcomplex

$$(\Delta^n \times \Delta^2)^{(S'_{i-1})}.$$

It follows that the inclusion

$$(\Delta^n \times \Delta^2)^{(S'_{i-1})} \subset (\Delta^n \times \Delta^2)^{(S'_i)}$$

is a sequence of inner anodyne extensions that is achieved by first attaching $d_0 Q_i$ and then attaching Q_i if $i < n$.

Thus, there is a sequence of inner anodyne extensions

$$(\Delta^n \times \Delta^2)^{(\emptyset)} \subset (\Delta^n \times \Delta^2)^{(S_0)} \subset (\Delta^n \times \Delta^2)^{(S'_i)}$$

if $i < n$. The "last" simplex Q_n has

$$d_i Q_n \in (\Delta^n \times \Delta^2)^{(S'_{n-1})}$$

if $i > 0$, and $d_0 Q_n$ is not a face any simplex in S'_{n-1} since it is a maximal non-degenerate simplex of $\Delta^n \times \Delta^1$.

The image of the 1-simplex $(0,0) \to (0,1)$ under the composite

$$\alpha_* : \partial\Delta^n \times \Delta^1 \xrightarrow{i\times 1} \Delta^n \times \Delta^1 \xrightarrow{\alpha} X$$

is a quasi-isomorphism of X, since that composite is a quasi-isomorphism of $\mathbf{hom}(\partial\Delta^n, X)$. It follows from Corollary 16 that the last of the extension problems (the dotted arrow) in the list

$$\begin{tikzcd}
(\Delta^n \times \Delta^2)^{(\emptyset)} \arrow[r] \arrow[d] & X \\
(\Delta^n \times \Delta^2)^{(S)} \arrow[ur] \arrow[d] & \\
(\Delta^n \times \Delta^2)^{(S'_{n-1})} \arrow[uur] \arrow[d] & \\
\Delta^n \times \Delta^2 \arrow[uuur, dotted] &
\end{tikzcd}$$

can be solved. ∎

Corollary 21. *Suppose that $i : A \to B$ is a cofibration of simplicial sets such that i is a bijection on vertices. Suppose that X is a quasi-category. Then the induced map*

$$i^* : \mathbf{hom}(B, X) \to \mathbf{hom}(A, X)$$

of quasi-categories creates quasi-isomorphisms.

Lemma 22. *Suppose that X is a quasi-category, and suppose given a diagram*

$$\begin{tikzcd}
(\Delta^n \times \{\epsilon\}) \cup (\partial\Delta^n \times \Delta^1) \arrow[r, "{(\beta,\alpha)}"] \arrow[d] & X \\
\Delta^n \times \Delta^1 \arrow[ur, dotted] &
\end{tikzcd}$$

where $\alpha : \partial\Delta^n \times \Delta^1 \to X$ is a quasi-isomorphism of $\mathbf{hom}(\partial\Delta^n, X)$, and ϵ is 0 or 1. Then the indicated extension problem can be solved.

Proof. We'll suppose that $\epsilon = 0$. The case $\epsilon = 1$ is similar (or even dual).

Write h_i for the non-degenerate $(n+1)$-simplex

$$\begin{tikzcd}
(0,0) \arrow[r] & \dots \arrow[r] & (i,0) \arrow[d] & & \\
& & (i,1) \arrow[r] & \dots \arrow[r] & (n,1)
\end{tikzcd}$$

Let $T = \{h_1, \dots, h_n\}$ and write $(\Delta^n \times \Delta^1)^{(T)}$ for the subcomplex of $\Delta^n \times \Delta^1$ generated by the subcomplex

$$(\Delta^n \times \Delta^1)^{(\emptyset)} = (\Delta^n \times \{0\}) \cup (\partial\Delta^n \times \Delta^1)$$

and the simplices in T. Then the inclusion

$$(\Delta^n \times \Delta^1)^{(\emptyset)} \subset (\Delta^n \times \Delta^1)^{(S_{n-1})}$$

is inner anodyne and there is a pushout diagram

$$\begin{tikzcd}
\Lambda^{n+1}_0 \arrow[r] \arrow[d] & (\Delta^n \times \Delta^1)^{(T)} \arrow[d] \\
\Delta^{n+1} \arrow[r, "h_0"'] & \Delta^n \times \Delta^1
\end{tikzcd}$$

In the diagram

$$\begin{tikzcd}
(\Delta^n \times \Delta^1)^{(\emptyset)} \arrow[r, "{(\beta,\alpha)}"] \arrow[d] & X \\
(\Delta^n \times \Delta^1)^{(T)} \arrow[ur, "\gamma"] \arrow[d] & \\
\Delta^n \times \Delta^1 \arrow[uur, dotted] &
\end{tikzcd}$$

the extension γ takes the simplex $(0,0) \to (0,1)$ to a quasi-isomorphism of X, so the indicated extension problem can be solved by Corollary 16. $\square$

Corollary 23. *Suppose that X is a quasi-category. Then the map*

$$i^* : J(\mathbf{hom}(\Delta^n, X)) \to J(\mathbf{hom}(\partial\Delta^n, X))$$

is a Kan fibration for $n \geq 0$.

Proof. The map $i^* : \mathbf{hom}(\Delta^n, X) \to \mathbf{hom}(\partial\Delta^n, X)$ is an inner fibration by Theorem 9, and has the path lifting property for quasi-isomorphisms by Lemma 22.

Lemma 19 says that i^* creates quasi-isomorphisms. It follows that the diagram

$$\begin{tikzcd}
J(\mathbf{hom}(\Delta^n, X)) \arrow[r] \arrow[d, "i^*"'] & \mathbf{hom}(\Delta^n, X) \arrow[d, "i^*"] \\
J(\mathbf{hom}(\partial\Delta^n, X)) \arrow[r] & \mathbf{hom}(\partial\Delta^n, X)
\end{tikzcd}$$

is a pullback. The map

$$i^* : J(\mathbf{hom}(\Delta^n, X)) \to J(\mathbf{hom}(\Delta^n, X))$$

is thus an inner fibration between Kan complexes that has the path lifting property, and is therefore a Kan fibration by Corollary 18. $\square$

Proposition 24. *Suppose that X is a quasi-category. Then the map*

$$\eta^* : \mathbf{hom}(I, X) \to \mathbf{hom}_I(\Delta^1, X)$$

is a trivial Kan fibration.

Proof. The lifting problem

$$\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[d] & \mathbf{hom}(I, X) \arrow[d, "\eta^*"] \\
\Delta^n \arrow[r, "\alpha"'] \arrow[ur, dotted] & \mathbf{hom}_I(\Delta^1, X)
\end{tikzcd}$$

is isomorphic to the lifting problem

$$\begin{tikzcd}
\Delta^1 \arrow[r, "\alpha_*"] \arrow[d, "\eta"'] & \mathbf{hom}(\Delta^n, X) \arrow[d, "i^*"] \\
I \arrow[r] \arrow[ur, dotted] & \mathbf{hom}(\partial\Delta^n, X)
\end{tikzcd} \tag{1}$$

The map α_* is a quasi-isomorphism by Lemma 19, and so the diagram (1) factors through the diagram

$$\begin{tikzcd}
\Delta^1 \arrow[r, "\alpha_*"] \arrow[d, "\eta"'] & J(\mathbf{hom}(\Delta^n, X)) \arrow[d, "i^*"] \\
I \arrow[r] \arrow[ur, dotted] & J(\mathbf{hom}(\partial\Delta^n, X))
\end{tikzcd}$$

The map i^* is a Kan fibration by Corollary 23. The map η is a trivial cofibration in the standard model structure for simplicial sets, so the lifting exists. $\square$

3 The quasi-category model structure

Let $I = B(\pi(\Delta^1))$ represent an interval theory on simplicial sets in the sense of [2]. In particular, recall that the assignment

$$\mathcal{P}(\underline{n}) \mapsto \square^n = I^{\times n}$$

defines a functor $I^\bullet : \square \to s\mathbf{Set}$, giving a representation of the box category $\square$ in simplicial sets.

The existence of two distinct maps $0, 1 : \Delta^0 \to I$ is part of the structure of an interval theory, and in this case these maps are given by the the two objects $0, 1$ of the groupoid $\pi(\Delta^1)$. An I-homotopy $f \sim_I g$ for an interval I is a commutative diagram

$$\begin{tikzcd}
K \arrow[d, "(1_K{,}0)"'] \arrow[dr, "f"] & \\
K \times I \arrow[r, "h"] & X \\
K \arrow[u, "(1_K{,}1)"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Write $\pi_I(K, X)$ for the corresponding set of I-homotopy classes of maps.

Following [2], write $\partial\square^n$ for the union of the faces

$$I^{\times(k-1)} \times \{\epsilon\} \times I^{\times(n-k)}$$

of $\square^n$ in the category of simplicial sets. The subcomplex $\sqcap^n_{(k,\epsilon)} \subset \partial\square^n$ is the result of deleting the face $I^{\times(k-1)} \times \{\epsilon\} \times I^{\times(n-k)}$ from $\partial\square^n$.

The set S of inner anodyne extensions $\Lambda^n_k \subset \Delta^n$ and the interval I together determine an (I, S)-model structure on the category $s\mathbf{Set}$ of simplicial sets [2], for which the cofibrations are the monomorphisms, and the fibrant objects are those simplicial sets Z for which the map $Z \to *$ has the right lifting property with respect all inclusions

$$(\Lambda^n_k \times \square^m) \cup (\Delta^n) \times \partial\square^m) \subset \Delta^n \times \square^m \tag{2}$$

induced by inner horns $\Lambda^n_k \subset \Delta^n$, and with respect to all maps

$$(\partial\Delta^n \times \square^m) \cup (\Delta^n \times \sqcap^m_{(k,\epsilon)}) \subset \Delta^n \times \square^m. \tag{3}$$

A weak equivalence of the (I, S)-model structure is a map $X \to Y$ which induces an isomorphism

$$\pi_I(Y, Z) \xrightarrow{\cong} \pi_I(X, Z)$$

in I-homotopy classes of maps for all fibrant objects Z.

There is a natural cofibration $j : X \to LX$, such that the map j is in the saturation of the set of maps described in (2) and (3), and LX has the right lifting property with respect to all such maps, and is therefore fibrant for the (I, S)-model structure. The map $j : X \to LX$ is a fibrant model for the (I, S)-model structure.

A map $X \to Y$ is a weak equivalence if and only if the induced map $LX \to LY$ is an I-homotopy equivalence.

If $h : K \times I \to X$ is an I-homotopy taking values in a quasi-category X then the composite map

$$K \times \Delta^1 \xrightarrow{1_K \times \eta} K \times I \xrightarrow{h} X$$

is a quasi-isomorphism of quasi-category $\mathbf{hom}(K, X)$: the requisite inverse and 2-simplices are defined in I. Following Joyal [5], let $\tau_0(K, X)$ denote the set of isomorphism classes in $P(\mathbf{hom}(K, X))$. It follows that there is an induced function

$$\pi_I(K, X) \xrightarrow{\eta^*} \tau_0(K, X).$$

We have, from Proposition 24, a trivial Kan fibration

$$\eta^* : \mathbf{hom}(I, \mathbf{hom}(K, X)) \to \mathbf{hom}_I(\Delta^1, \mathbf{hom}(K, X)).$$

The vertices of the space $\mathbf{hom}_I(\Delta^1, \mathbf{hom}(K, X))$ are the quasi-isomorphisms of the quasi-category $\mathbf{hom}(K, X)$. The trivial fibration η^* is, among other things, surjective on vertices, and so for every quasi-isomorphism $h : K \times \Delta^1 \to X$ there is an extension

$$\begin{tikzcd} K \times \Delta^1 \arrow[r, "h"] \arrow[d, "1_K \times \eta"'] & X \\ K \times I \arrow[ur, dotted, "H"'] & \end{tikzcd}$$

Thus if there is a homotopy $h : K \times \Delta^1 \to X$ from f to g that is a quasi-isomorphism of $\mathbf{hom}(K, X)$, then there is an I-homotopy $H : K \times I \to X$ from f to g, and conversely. We have proven the following:

Proposition 25. *Suppose that X is a quasi-category and that K is a simplicial set. Then precomposing with the map $\eta : \Delta^1 \to I$ defines a bijection*

$$\pi_I(K, X) \cong \tau_0(K, X).$$

This bijection is natural in simplicial sets K and quasi-categories X.

Lemma 26. *Suppose that X is a quasi-category. Then all extension problems*

$$\begin{tikzcd} (\partial\Delta^m \times \square^n) \cup (\Delta^m \times \sqcap^n_{(i,\epsilon)}) \arrow[r] \arrow[d] & X \\ \Delta^m \times \square^n \arrow[ur, dotted] & \end{tikzcd}$$

can be solved for $\epsilon = 0, 1$.

Proof. The extension problem in the statement of the Lemma can be rewritten as a lifting problem

$$\begin{tikzcd} \sqcap^n_{(i,\epsilon)} \arrow[r, "\alpha"] \arrow[d, "j"'] & \mathbf{hom}(\Delta^m, X) \arrow[d, "i^*"] \\ \square^n \arrow[r, "\beta"'] \arrow[ur, dotted] & \mathbf{hom}(\partial\Delta^m, X) \end{tikzcd} \tag{4}$$

by adjointness. The maps α and β both map 1-simplices to quasi-isomorphisms, so the lifting problem (4) factors through a lifting problem

$$\begin{tikzcd} \sqcap^n_{(i,\epsilon)} \arrow[r, "\alpha"] \arrow[d, "j"'] & J(\mathbf{hom}(\Delta^m, X)) \arrow[d, "i^*"] \\ \square^n \arrow[r, "\beta"'] \arrow[ur, dotted] & J(\mathbf{hom}(\partial\Delta^m, X)) \end{tikzcd}$$

The map i^* in the diagram is a Kan fibration by Corollary 23, and the inclusion $\sqcap^n_{(i,\epsilon)} \subset \square^n$ is a standard weak equivalence (of contractible spaces), so that the dotted arrow exists. $\square$

Theorem 27. *Quasi-categories are the fibrant objects for the (I, S)-model structure on simplicial sets that is determined by the set S of inner anodyne extensions and the interval theory defined by the space $I = B(\pi(\Delta^1))$.*

Proof. If the inclusion $A \subset B$ is inner anodyne, then so are all inclusions

$$(A \times \square^n) \cup (B \times \partial\square^n) \subset (B \times \square^n).$$

This is a consequence of Theorem 9.

If X is a quasi-category, then the map $X \to *$ has the right lifting property with respect to all inclusions

$$(\partial\Delta^m \times \square^n) \cup (\Delta^m \times \sqcap^n_{(i,\epsilon)}) \subset \Delta^m \times \square^n$$

by Lemma 26. □

Proposition 25 says that the naive homotopy classes of maps $\pi_I(K, X)$ taking values in a quasi-category X for the (I, S)-model structure for simplicial sets coincide up to natural bijection with Joyal's set

$$\tau_0(K, X) = \pi_0 J(\mathbf{hom}(K, X)),$$

so that the fibrant objects, weak equivalences and cofibrations of the (I, S)-structure coincide with the respective classes of maps in Joyal's model structure for quasi-categories [5]. It follows that the (I, S)-structure for simplicial sets coincides with Joyal's structure.

In particular, the weak equivalences and the fibrations for the (I, S)-structure are the *weak categorical equivalences* and the *pseudo-fibrations* of [5], respectively. We shall continue to use these terms.

In particular, a weak categorical equivalence is a simplicial set map $f : X \to Y$ such that the induced map

$$f^* : \pi_I(Y, Z) \to \pi_I(X, Z)$$

is a bijection for all quasi-categories Z.

Observe that a map $p : X \to Y$ is a weak categorical equivalence and a pseudo-fibration if and only if it is a trivial fibration in the standard model structure for simplicial sets.

The natural fibrant model $j_X : X \to LX$ for the (I, S)-model structure also has a much simpler construction, such that j is an inner anodyne extension and LX is a quasi-category.

Lemma 28. *If $g : X \to Y$ is a categorical weak equivalence, then the induced map $g_* : P(X) \to P(Y)$ is an equivalence of categories.*

Proof. There is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "j_X"] \arrow[d, "g"'] & LX \arrow[d, "g_*"] \\ Y \arrow[r, "j_Y"'] & LY \end{tikzcd}$$

where j_X and j_Y are inner anodyne extensions and LX and LY are quasi-categories. The map g_* is a categorical weak equivalence of quasi-categories, and is therefore an I-homotopy equivalence. It follows that the induced map

$$g_* : P(LX) \to P(LY)$$

is an equivalence of categories.

In effect, the functor $X \mapsto P(X)$ preserves finite products and $P(I) \cong \pi(\Delta^1)$. Finally, Lemma 6 implies that the inner anodyne extensions i_X, i_Y induce isomorphisms $P(X) \cong P(X_f)$ and $P(Y) \cong P(Y_f)$. □

We also have the following, by a very similar argument:

Lemma 29. *Every categorical weak equivalence is a standard weak equivalence of simplicial sets.*

Proof. The inner horn inclusions $\Lambda^n_k \subset \Delta^n$ are standard weak equivalences, so that the inner anodyne extension $j_X : X \to LX$ is a standard weak equivalence. The interval I is contractible in the standard model structure, and a map $f : Z \to W$ of quasi-categories is a weak equivalence if and only if it is an I-homotopy equivalence, so that every I-homotopy equivalence is a standard weak equivalence. □

Lemma 30. *Suppose that $g : X \to Y$ is a categorical weak equivalence, and that K is a simplicial set. Then the map $g \times 1_K : X \times K \to Y \times K$ is a categorical weak equivalence.*

Proof. Suppose that Z is a quasi-category. The exponential law induces a natural bijection

$$\pi_I(X, \mathbf{hom}(K, Z)) \cong \pi_I(X \times K, Z).$$

The function complex $\mathbf{hom}(K, Z)$ is a quasi-category by Corollary 11, so that the induced map

$$g^* : \pi_I(Y, \mathbf{hom}(K, Z)) \to \pi_I(X, \mathbf{hom}(K, Z))$$

is a bijection. □

Corollary 31. *Suppose that $i : A \to B$ is a cofibration and a categorical weak equivalence, and that $j : C \to D$ is a cofibration. Then the cofibration*

$$(B \times C) \cup (A \times D) \subset B \times D$$

is a categorical weak equivalence.

Example 32. A functor $f : C \to D$ of small categories induces a categorical weak equivalence $f_* : BC \to BD$ if and only if the finctor f is an equivalence of categories.

In effect, BC and BD are quasicategories, so that f_* is a categorical weak equivalence if and only if it is an I-homotopy equivalence. This means that there is a functor $g : D \to C$ and homotopies $C \times \pi(\Delta^1) \to C$ and $D \times \pi(\Delta^1) \to D$ which define the homotopies $g \cdot f \simeq 1$ and $f \cdot g \simeq 1$.

It follows that the quasi-category BC is weakly equivalent to a point if and only if C is a trivial groupoid.

Example 33. Suppose that C is a small category. Then JBC is the nerve $B(Iso(C))$ of the nerve of the groupoid of isomorphisms in C. The inclusion $JBC \subset BC$ is a quasi-category equivalence if and only if C is a groupoid.

Thus, in general, the map $JX \subset X$ is not a quasi-category equivalence for quasi-categories X.

Lemma 34. *1) Suppose that $p : X \to Y$ is a pseudo-fibration of quasi-categories. Then the induced map $p : J(X) \to J(Y)$ is a Kan fibration.*

2) Suppose that $p : X \to Y$ is a trivial fibration of quasi-categories. Then the induced map $p : J(X) \to J(Y)$ is a trivial Kan fibration.

3) Suppose that $f : X \to Y$ is a categorical weak equivalence of quasi-categories. Then the induced map $f : J(X) \to J(Y)$ is a weak equivalence of simplicial sets.

Proof. For statement 1), suppose given a commutative diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & J(X) \arrow[r] \arrow[d, "\theta"'] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[urr, dotted] & J(Y) \arrow[r] & Y \end{tikzcd}$$

where $0 < k < n$. Then the dotted arrow θ exists since p is an inner fibration. The map $P(\Lambda^n_k) \to P(\Delta^n)$ is an isomorphism, so θ maps all 1-simplices of Δ^n to quasi-isomorphisms. Thus, θ factors through a map $\theta' : \Delta^n \to J(X)$, and so the map $p : J(X) \to J(Y)$ is an inner fibration.

The map p has the path lifting property in the sense that all lifting problems

$$\begin{tikzcd} \{\epsilon\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ I \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

can be solved, where $\epsilon = 0, 1$. Any lift $I \to X$ has its image in $J(X)$, since I is a Kan complex.

Suppose given a lifting problem

$$\begin{tikzcd} \{\epsilon\} \arrow[r] \arrow[d] & J(X) \arrow[d, "p"] \\ \Delta^1 \arrow[r, "\alpha"'] \arrow[ur, dotted] & J(Y) \end{tikzcd}$$

Then there is a morphism $\alpha' : I \to J(Y)$ such that $\alpha' \cdot \eta = \alpha$, where $\eta : \Delta^1 \to I$ is the trivial cofibration that we've been using, since $J(Y)$ is a Kan complex. It follows that there is a commutative diagram

$$\begin{tikzcd}
\{\epsilon\} \arrow[rr] \arrow[d] & & J(X) \arrow[r] \arrow[d, "p"] & X \arrow[d, "p"] \\
\Delta^1 \arrow[r] & I \arrow[r, "\alpha'"'] \arrow[ur] & J(Y) \arrow[r] & Y
\end{tikzcd}$$

and so the map $p : J(X) \to J(Y)$ has the path lifting property. The map p is therefore a Kan fibration, by Corollary 18.

For statement 2), the trivial fibration $p : X \to Y$ creates quasi-isomorphisms, so that the diagram

$$\begin{tikzcd}
J(X) \arrow[r] \arrow[d, "p"'] & X \arrow[d, "p"] \\
J(Y) \arrow[r] & Y
\end{tikzcd}$$

is a pullback. It follows that the map $p : J(X) \to J(Y)$ is a trivial Kan fibration.

Statement 3) is a formal consequence of statement 2), by the usual factorization trick: the map f is a composite $f = q \cdot j$ where q is a pseudo-fibration and a categorical weak equivalence (a trivial fibration), and j is a section of a trivial fibration. $\square$

Remark 35. Suppose that $f : X \to Y$ is a map of quasi-categories such that $f_* : J(X) \to J(Y)$ is a weak equivalence of simplicial sets. It does not follow that f is a categorical weak equivalence.

For example, suppose that C is a small category and recall that the core $J(BC)$ of the nerve of C is the nerve $B(\mathrm{Iso}(C))$ of the groupoid of isomorphisms of C. The map $B(\mathrm{Iso}(C)) \to BC$ induces an isomorphism of cores, but is not a categorical weak equivalence in general, since C may not be a groupoid. See also Example 33.

The fibrant model construction $j : X \to LX$ for the quasi-category model structure is defined by a countable sequence of cofibrations

$$X = X_0 \to X_1 \to X_2 \to \dots$$

with $LX = \varinjlim_i X_i$. In all cases, X_{i+1} is constructed from X_i by forming the pushout

$$\begin{tikzcd}
\bigsqcup_{\Lambda^n_k \to X_i} \Lambda^n_k \arrow[r] \arrow[d] & X_i \arrow[d] \\
\bigsqcup_{\Lambda^n_k \to X_i} \Delta^n \arrow[r] & X_{i+1}
\end{tikzcd}$$

where the disjoint union is indexed over all maps $\Lambda^n_k \to X_i$ of inner horns to X_i.

Suppose that α is a regular cardinal. It is a consequence of the construction that if X is α-bounded, then LX is α-bounded. Each of the functors $X \mapsto X_i$ preserves monomorphisms and filtered colimits, and it follows that the functor $X \mapsto LX$ has these same properties. If A and B are subobjects of a simplicial set X, then there is an isomorphism

$$L(A \cap B) \xrightarrow{\cong} LA \cap LB,$$

as subobjects of LX.

Lemma 36. *Suppose that $i : X \to Y$ is a cofibration and a categorical weak equivalence. Suppose that $A \subset Y$ is an α-bounded subobject of Y. Then there is an α-bounded subobject B of Y with $A \subset B$, such that the map $B \cap X \to B$ is a categorical weak equivalence.*

This result is a consequence of the method of proof of Lemma 4.9 of [2].

Proof. The map $i_* : LX \to LY$ is a filtered colimit of the maps $L(B \cap X) \to LB$, indexed over the α-bounded subobjects B of Y. All diagrams

$$\begin{tikzcd} L(B \cap X) \arrow[r] \arrow[d] & LX \arrow[d, "i_*"] \\ LB \arrow[r] & LY \end{tikzcd}$$

are pullbacks. Every map $f : Z \to W$ between quasi-categories has a functorial factorization $f = p \cdot j$, where p is a pseudo-fibration and j is a section of a trivial fibration. For the map $LX \to LY$ this factorization has the form

$$\begin{tikzcd} LX \arrow[r, "j"] \arrow[dr] & Z \arrow[d, "p"] \\ & LY \end{tikzcd} \tag{5}$$

For each α-bounded subobject $B \subset Y$, the factorization of the map $L(B \cap X) \to LB$ can be written

$$\begin{tikzcd} L(B \cap X) \arrow[r, "j_B"] \arrow[dr] & Z_B \arrow[d, "p_B"] \\ & LB \end{tikzcd} \tag{6}$$

The diagram (5) is a filtered colimit of diagrams (6). It follows that all lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & Z_A \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & LA \end{tikzcd}$$

have solutions over some LB_1, where B_1 is an α-bounded subcomplex of Y such that $A \subset B_1$. Continue inductively, to produce a chain of α-bounded subobjects

$$A \subset B_1 \subset B_2 \subset \dots$$

such that all lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & Z_{B_i} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & LB_i \end{tikzcd}$$

have solutions over LB_{i+1}.

Set $B = \varinjlim_i B_i$. Then the map $p_B : Z_B \to LB$ is a trivial fibration, so the map $L(B \cap X) \to LB$ is a quasi-category weak equivalence. $\square$

Lemma 37. *Suppose that $f : X \to Y$ is a categorical weak equivalence of quasi-categories and that A is a simplicial set. Then the map*

$$\mathbf{hom}(A, X) \to \mathbf{hom}(A, Y)$$

is a categorical weak equivalence of quasi-categories.

Proof. The object $\mathbf{hom}(A, X)$ is a quasi-category if X is a quasi-category, by Corollary 11.

The map f has a factorization $f = q \cdot j$, where q is a trivial fibration and j is a section of a trivial fibration. The functor $X \mapsto \mathbf{hom}(A, X)$ preserves trivial fibrations, and therefore preserves categorical weak equivalences between quasi-categories. $\square$

Proposition 38. *A map $f : X \to Y$ between quasi-categories is a categorical weak equivalence if and only if it induces equivalences of groupoids*

$$\begin{aligned} &f_* : \pi J(\mathbf{hom}(\partial\Delta^n, X)) \to \pi J(\mathbf{hom}(\partial\Delta^n, Y)) \textit{ and} \\ &f_* : \pi J(\mathbf{hom}(\Delta^n, X)) \to \pi J(\mathbf{hom}(\Delta^n, Y)) \end{aligned} \tag{7}$$

for $n \geq 1$.

Proof. Suppose that $f : X \to Y$ is a quasi-weak equivalence of quasi-categories. Then all induced maps

$$\mathbf{hom}(A, X) \to \mathbf{hom}(A, Y)$$

are quasi-weak equivalences of quasi-categories by Lemma 37. The induced maps

$$J\mathbf{hom}(A, X) \to J\mathbf{hom}(A, Y)$$

are weak equivalences of Kan complexes by Lemma 34, and therefore induce equivalences of fundamental groupoids

$$\pi J\mathbf{hom}(A, X) \to \pi J\mathbf{hom}(A, Y)$$

It follows that the maps of (7) are equivalences of groupoids.

For the converse, suppose that all morphisms (7) are weak equivalences of groupoids.

It suffices to assume that f is a pseudo-fibration, by the usual fibration replacement trick for maps between Kan complexes: if $f = p \cdot i$ where i is a section of a trivial fibration and p is a pseudo-fibration, then f satisfies the conditions of the Lemma if and only if p does so.

Suppose, therefore, that f is a pseudo-fibration. We want to solve the lifting problem

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ \Delta^n \arrow[r, "\beta"'] \arrow[ur, dotted] & Y \end{tikzcd}$$

The map

$$f_* : \mathbf{hom}(\Delta^n, X) \to \mathbf{hom}(\Delta^n, Y)$$

is a pseudo-fibration by Lemma 30. The map

$$f_* : \pi J(\mathbf{hom}(\Delta^n, X)) \to \pi J(\mathbf{hom}(\Delta^n, Y))$$

is an equivalence of groupoids by assumption, so that there is a quasi-isomorphism $\Delta^1 \to \mathbf{hom}(\Delta^n, Y)$ from the vertex β to $p(\gamma)$ for some $\gamma : \Delta^n \to X$.

All pseudo-fibrations $p : Z \to W$ satisfy the path lifting property

$$\begin{tikzcd} \{\epsilon\} \arrow[r] \arrow[d] & Z \arrow[d, "p"] \\ I \arrow[r] \arrow[ur, dotted] & W \end{tikzcd}$$

It follows that all pseudo-fibrations p between quasi-categories satisfy a lifting property

$$\begin{tikzcd} \{\epsilon\} \arrow[r] \arrow[d] & Z \arrow[d, "p"] \\ \Delta^1 \arrow[r, "\gamma"'] \arrow[ur, dotted, "\theta"] & W \end{tikzcd}$$

for quasi-isomorphisms γ, since a quasi-isomorphism γ extends to a morphism

$$I \to J(W) \subset W.$$

We can further assume that the lifting θ is a quasi-isomorphism of Z.

It follows that there is a quasi-isomorphism $\Delta^1 \to \mathbf{hom}(\Delta^n, X)$ from a pre-image β' of β to γ, and in particular there is a lifting

$$\begin{tikzcd} & X \arrow[d, "f"] \\ \Delta^n \arrow[r, "\beta"'] \arrow[ur, "\beta'"] & Y \end{tikzcd}$$

The map
$$f_* : J(\mathbf{hom}(\partial\Delta^n, X)) \to J(\mathbf{hom}(\partial\Delta^n, Y))$$
is a Kan fibration by Lemma 34. The vertices $\beta' \cdot i$ and α of the Kan complex $J(\mathbf{hom}(\partial\Delta^n, X))$ have the same image, namely $\beta \cdot i$ under this fibration f_*, and are therefore in the fibre $F_{\beta\cdot i}$ of f_* over $\beta \cdot i$. The fibration f_* induces an equivalence of fundamental groupoids by assumption, so that the fibre $F_{\beta\cdot i}$ is a connected Kan complex. It follows that there is a path $h : \Delta^1 \to F_{\beta\cdot i}$ from α to $\beta' \cdot i$, and so there is a map h' making the diagram

$$\begin{tikzcd}
\Delta^1 \arrow[r, "h"] \arrow[d, "\eta"'] & F_{\beta\cdot i} \\
I \arrow[ur, "h'"'] &
\end{tikzcd}$$

since the map η is a trivial cofibration in the standard model structure for simplicial sets. Write H for the adjoint of the composite

$$I \xrightarrow{h'} F_{\beta\cdot i} \to J(\mathbf{hom}(\partial\Delta^n, X).$$

Then the lifting problem

$$\begin{tikzcd}
(\Delta^n \times \{1\}) \cup (\partial\Delta^n \times I) \arrow[rr, "{(\beta', H)}"] \arrow[d] & & X \arrow[d, "f"] \\
\Delta^n \times I \arrow[rr] \arrow[urr, dotted, "\theta"] & & Y
\end{tikzcd}$$

can be solved since f is a pseudo-fibration, by Lemma 22. It follows that there is a commutative diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
\Delta^n \arrow[r, "\beta"'] \arrow[ur, "\theta'"] & Y
\end{tikzcd}$$

where θ' is the composite

$$\Delta^n \times \{0\} \subset \Delta^n \times I \xrightarrow{\theta} X.$$

□

Corollary 39. *A map $f : X \to Y$ between Kan complexes is a standard weak equivalence of simplicial sets if and only if it induces equivalences of groupoids*

$$f_* : \pi(\mathbf{hom}(\partial\Delta^n, X)) \to \pi(\mathbf{hom}(\partial\Delta^n, Y)) \tag{8}$$

for $n \geq 1$.

Proof. Suppose that all maps (8) are weak equivalences of groupoids. Then the morphism $\pi(X) \to \pi(Y)$ is an equivalence of fundamental groupoids, since it is a retract of the equivalence

$$\pi\mathbf{hom}(\partial\Delta^1, X) \to \pi\mathbf{hom}(\partial\Delta^1, Y).$$

Any vertex $* \to \Delta^n$ induces a natural weak equivalence

$$\mathbf{hom}(\Delta^n, X) \xrightarrow{\simeq} X$$

of Kan complexes, while there is an identification

$$J(\mathbf{hom}(\Delta^n, X)) = \mathbf{hom}(\Delta^n, X).$$

It follows that all induced groupoid morphisms

$$\pi J(\mathbf{hom}(\Delta^n, X)) \to \pi J(\mathbf{hom}(\Delta^n, Y))$$

are equivalences.

It follows from Proposition 38 that the map $f : X \to Y$ is a categorical weak equivalence. The Kan complexes X and Y are quasi-categories, so Lemma 29 implies that f is a standard weak equivalence. $\square$

Corollary 39 can also be proved directly, by using traditional methods of simplicial homotopy theory.

It follows from Lemma 4.13 of [2] that a map $p : X \to Y$ between quasi-categories is a pseudo-fibration if and only if it has the right lifting property with respect to the maps (2) and (3).

The maps (2) induce isomorphisms of path categories, and the map (3) induces an isomorphism of path categories

$$P((\partial\Delta^n \times \square^m) \cup (\Delta^n \times \sqcap^m_{(k,\epsilon)})) \xrightarrow{\cong} P(\Delta^n \times \square^m)$$

if $m \geq 1$. It follows that a functor $p : C \to D$ between small categories induces a pseudo-fibration if and only if it has the right lifting property with respect to all functors $\mathbf{n} \times \{\epsilon\} \to \mathbf{n} \times \pi(\Delta^1)$, where $\epsilon = 0, 1$.

It is then an exercise to show that the functor $p : C \to D$ defines a pseudo-fibration $BC \to BD$ if and only if it has the *isomorphism lifting property* in the sense that all lifting problems

$$\begin{tikzcd} \{\epsilon\} \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ \mathbf{1} \arrow[r, "\alpha"'] \arrow[ur, dotted] & D \end{tikzcd}$$

have solutions, where $\epsilon = 0, 1$, and the morphism defined by the functor α is an isomorphism of D.

If $p : C \to D$ has the isomorphism lifting property, then the induced functors $p : C^{\mathbf{n}} \to D^{\mathbf{n}}$ have the isomorphism lifting property, for all $n \geq 1$.

Example 40. Suppose that the functor $p : C \to D$ has the isomorphism lifting property, and suppose that the induced map $p_* : BC \to BD$ of quasi-categories satisfies the criteria for a categorical weak equivalence that are given by Proposition 38. These criteria mean, precisely, that all induced functors

$$\mathrm{Iso}(C^{\mathbf{n}}) \to \mathrm{Iso}(D^{\mathbf{n}}) \tag{9}$$

and

$$\mathrm{Iso}(C^{P(\partial\Delta^n)}) \to \mathrm{Iso}(D^{P(\partial\Delta^n)}) \tag{10}$$

are equivalences of groupoids for $n \geq 0$. These functors are then trivial fibrations of groupoids in the traditional sense, by Lemma 34.

To solve the lifting problems

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & BC \arrow[d, "p_*"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & BD \end{tikzcd}$$

it suffices to assume that $n \leq 2$, since the induced functors $P(\partial\Delta^n) \to P(\Delta^n)$ are isomorphisms for $n \geq 3$.

Every object x of D is isomorphic to the image $p(y)$ of some object y of C since the map $\mathrm{Iso}(C) \to \mathrm{Iso}(D)$ is an equivalence of groupoids. We make a specific choice of isomorphism $\alpha : x \xrightarrow{\cong} p(y)$ in D. The lifting problem

$$\begin{tikzcd} \{1\} \arrow[r, "y"] \arrow[d] & C \arrow[d, "p"] \\ \mathbf{1} \arrow[r, "\alpha"'] \arrow[ur, dotted] & D \end{tikzcd}$$

has a solution, so there is an object z of C such that $p(z) = x$.

Suppose given a lifting problem

$$\begin{tikzcd} \partial\Delta^2 \arrow[r, "\alpha"] \arrow[d] & BC \arrow[d, "p_*"] \\ \Delta^2 \arrow[r, "\beta"'] \arrow[ur, dotted] & BD \end{tikzcd}$$

The map $\mathrm{Iso}(C^{\mathbf{2}}) \to \mathrm{Iso}(D^{\mathbf{2}})$ is an equivalence of groupoids, so there is an isomorphism $h : \mathbf{1} \to \mathrm{Iso}(D^{\mathbf{2}})$ from β to a functor $p(\gamma)$, for some $\gamma : \mathbf{2} \to C$. The isomorphism h lifts along the pseudo-fibration $C^{\mathbf{2}} \to D^{\mathbf{2}}$ to an isomorphism $H : \omega \xrightarrow{\cong} \gamma$ in $C^{\mathbf{2}}$. The restriction $\omega|_{\partial\Delta^2}$ and α have the same image, namely $\beta|_{\partial\Delta^2}$ under p, and the map $p : \mathrm{Iso}(C^{P(\partial\Delta^2)}) \to \mathrm{Iso}(D^{P(\partial\Delta^2)})$ is a trivial fibration of groupoids. It follows that there is an isomorphism $\omega|_{\partial\Delta^2} \to \alpha$ of $\mathrm{Iso}(C^{P(\partial\Delta^2)})$ which maps to the identity of $\beta|_{\partial\Delta^2}$ under p. It follows that there is a (unique) functor $\zeta : \mathbf{2} \to C$ which maps to β under p and restricts to α on $\partial\Delta^2$.

The solution of the lifting problem

$$\begin{tikzcd} \partial\Delta^1 \arrow[r, "\alpha"] \arrow[d] & BC \arrow[d, "p_*"] \\ \Delta^1 \arrow[r, "\beta"'] \arrow[ur, dotted] & BD \end{tikzcd}$$

is very similar.

The moral is that a functor $f : C \to D$ is an equivalence of categories if and only if the functors (9) and (10) are equivalences of groupoids for $0 \leq n \leq 2$.

4 Products of simplices

Lemma 41. *Suppose that $\sigma : \Delta^n \to \Delta^r \times \Delta^s$ is a non-degenerate simplex. Then $n \leq r + s$.*

Proof. Suppose that σ is defined by the path

$$(i_0, j_0) \to (i_1, j_1) \to \cdots \to (i_n, j_n)$$

in the product poset $\mathbf{r} \times \mathbf{s}$. The simplex σ is non-degenerate, so there are no repeats in the string. Thus, $i_0 < i_1$ or $j_0 < j_1$.

If $i_0 < i_1$ then the string

$$d_0(\sigma) : \ (i_1, j_1) \to \cdots \to (i_n, j_n)$$

lies in a subobject of $\mathbf{r} \times \mathbf{s}$ isomorphic to a poset $\mathbf{r}' \times \mathbf{s}$, where $r' < r$, and inductively $d_0(\sigma)$ has length $L(d_0(\sigma))$ bounded above by $r' + s$. Thus, σ has length

$$n = 1 + L(d_0(\sigma)) \leq 1 + (r' + s) \leq r + s.$$

The same outcome obtains (with a similar argument) if $j_0 < j_1$. ∎

Corollary 42. *The non-degenerate simplices $\sigma : \Delta^n \to \Delta^r \times \Delta^s$ of maximal dimension have dimension $n = r + s$.*

Suppose that

$$\sigma : (0,0) = (i_0, j_0) \to \cdots \to (i_n, j_n) = (r, s) \tag{11}$$

is a non-degenerate simplex of maximal dimension $n = r+s$. Then $i_{k+1} \leq i_k + 1$ for $k < r$, for otherwise, there is a non-degenerate path

$$(i_k, j_k) \to (i_k + 1, j_k) \to (i_{k+1}, j_{k+1})$$

having the path $(i_k, j_k) \to (i_{k+1}, j_{k+1})$ as a face, and σ does not have maximal length. Similarly (or dually), $j_{k+1} \leq j_k + 1$ for $k < r$. Observe also that

a) if $i_{k+1} = i_k$ then $j_k < j_{k+1}$ so that $j_k + 1 = j_{k+1}$, and

b) (dually) if $j_{k+1} = j_k$ then $i_{k+1} = i_k + 1$.

Suppose that

$$\gamma : (0,0) = (i'_0, j'_0) \to \cdots \to (i'_n, j'_n) = (r,s)$$

is another simplex of maximal dimension $n = r+s$ in $\Delta^r \times \Delta^s$. Say that $\sigma \leq \gamma$ if $i_k \leq i'_k$ for $0 \leq k \leq n$.

Lemma 43. *For the ordering $\sigma \leq \gamma$ on the set of non-degenerate simplices of $\Delta^r \times \Delta^s$ of maximal dimension, the simplex*

$$(0,0) \to (0,1) \to \cdots \to (0,s) \to (1,s) \to \cdots \to (r,s)$$

is minimal, and the simplex

$$(0,0) \to (1,0) \to \cdots \to (r,0) \to (r,1) \to \cdots \to (r,s)$$

is maximal.

Proof. If σ is a path as (11), then there are relations

$$0 = i_0 \leq i_1 \leq \cdots \leq i_r$$

and $i_{k+1} \leq i_k + 1$. It follows that $0 \leq i_j \leq j$ for $0 \leq j \leq r$. Also, $i_j \leq r$ for all j, and hence for all $j > r$. It follows that the indicated simplex is maximal.

The relation

$$i_k + j_k = k,$$

holds for all non-degenerate simplices of maximal dimension — this is a consequence of the observations in statements a) and b) above. Thus, for $0 \leq k \leq r$, $i_{s+k} + j_{s+k} = s+k$ and $j_{s+k} \leq s$ together force $i_{s+k} = (s - j_{s+k}) + k \geq k$. The assertion that the indicated simplex is minimal follows. $\square$

Suppose that σ is a non-degenerate $(r+s)$-simplex such that the path defining σ contains a segment

$$\begin{tikzcd} & (i_k+1, j_k+1) \\ (i_k, j_k) \arrow[r] & (i_k+1, j_k) \arrow[u] \end{tikzcd}$$

Then the path σ' that is obtained from σ by replacing the segment above by the path

$$\begin{tikzcd} (i_k, j_k+1) \arrow[r] & (i_k+1, j_k+1) \\ (i_k, j_k) \arrow[u] \end{tikzcd} \tag{12}$$

satisfies $\sigma' \leq \sigma$.

Observe that $d_{k+1}(\sigma) = d_{k+1}(\sigma')$, and that σ and σ' are the only two non-degenerate $(r+s)$-simplices for which $d_{k+1}(\sigma)$ could be a face. In particular, if $\tau < \sigma'$ then $d_{k+1}(\sigma)$ is not a face of τ.

Lemma 44. *Suppose that σ is a non-degenerate $(r+s)$-simplex of $\Delta^r \times \Delta^s$.*

1) If σ is not maximal, then it contains a segment

$$\begin{tikzcd} (i_k, j_k+1) \arrow[r] & (i_k+1, j_k+1) \\ (i_k, j_k) \arrow[u] & \end{tikzcd}$$

2) If σ is not minimal, then it contains a segment

$$\begin{tikzcd} & (i_r+1, j_r+1) \\ (i_r, j_r) \arrow[r] & (i_r+1, j_r) \arrow[u] \end{tikzcd}$$

Proof. We prove statement 1). The proof of statement 2) is similar.

We argue by induction on $r + s \geq 2$.

If $r = s = 1$ there are two non-degenerate 2-simplices in $\Delta^1 \times \Delta^1$ and the one that is not maximal has the form (12).

Suppose that i_k is minimal such that $(i_k, j_k) = (i_k, s)$. If $i_k < r$, then σ has a segment

$$\begin{tikzcd} (i_k, r) \arrow[r] & (i_k+1, r) \\ (i_k, r-1) \arrow[u] & \end{tikzcd}$$

Suppose that $i_k = r$. Choose the minimal i_p such that $(i_p, j_p) = (r, j_p)$. Then $i_p > 0$ since σ is not maximal, and the segment of σ ending at $(r, j_p))$ has the form

$$(0,0) \to \cdots \to (r-1, j_p) \to (r, j_p).$$

This segment defines a maximal non-degenerate simplex of $\Delta^r \times \Delta^{j_p}$, which is susceptible to the argument of the first paragraph. This simplex therefore has a segment of the required form, as does the simplex σ. $\square$

An *interior simplex* of $\Delta^r \times \Delta^s$ is a poset morphism

$$\mathbf{m} \xrightarrow{\theta} \mathbf{r} \times \mathbf{s}$$

such that the composites

$$\mathbf{m} \xrightarrow{\theta} \mathbf{r} \times \mathbf{s} \to \mathbf{r}$$

$$\mathbf{m} \xrightarrow{\theta} \mathbf{r} \times \mathbf{s} \to \mathbf{s}$$

with the respective projections are surjective. Such a simplex θ cannot lie in the boundary subcomplex

$$(\partial\Delta^r \times \Delta^s) \cup (\Delta^r \times \partial\Delta^s),$$

for then one of the two composites above would fail to be surjective.

Theorem 45. *Suppose that* $0 < k < n$*. Then the inclusion*

$$(\Lambda^n_k \times \Delta^m) \cup (\Delta^n \times \partial\Delta^m) \subset \Delta^n \times \Delta^m$$

is inner anodyne.

Proof. Suppose that T is a set of non-degenerate $(m+n)$-simplices of $\Delta^m \times \Delta^n$ that is order closed in the sense that if $\gamma \in T$ and $\tau \leq \gamma$ then $\tau \in T$.

Given T pick a smallest σ such that $\sigma \notin T$. Then the collection of all simplices τ such that $\tau < \sigma$ is contained in T and the set $T' = T \cup \{\sigma\}$ is order closed. The empty set $\emptyset$ of non-degenerate $(m+n)$-simplices is the minimal order-closed set, and the full set of non-degenerate $(m+n)$-simplices is the maximal order-closed set.

Write

$$(\Delta^n \times \Delta^m)^{(T)}$$

for the subcomplex of $\Delta^m \times \Delta^n$ that is generated by the subcomplex

$$(\Delta^n \times \Delta^m)^{(\emptyset)} = (\Lambda^n_k \times \Delta^m) \cup (\Delta^n \times \partial\Delta^m)$$

and the simplices in T.

The idea of the proof is to show that all inclusions

$$(\Delta^n \times \Delta^m)^{(T)} \subset (\Delta^n \times \Delta^m)^{(T')}$$

are inner anodyne for order closed sets T and $T' = T \cup \{\sigma\}$ as above.

Every non-degenerate $(m+n)$-simplex σ has $d_0(\sigma)$ and $d_{m+n}(\sigma)$ in the subcomplex $(\Delta^n \times \Delta^m)^{(\emptyset)}$:

1) If the first member in the path

$$\sigma : \ (0,0) = (i_0, j_0) \to \cdots \to (i_{m+n}, j_{m+n}) = (n,m)$$

is the morphism $(0,0) \to (1,0)$ then $d_0(\sigma)$ is in the image of the poset morphism $d^0 \times 1 : (\mathbf{n-1}) \times \mathbf{m} \to \mathbf{n} \times \mathbf{m}$, which is in $\Lambda^n_k \times \Delta^m$ since $k \neq 0$. If the first member of the path σ is the morphism $(0,0) \to (0,1)$ then $d_0(\sigma)$ is in the image of the morphism $1 \times d^0 : \mathbf{n} \times (\mathbf{m-1}) \to \mathbf{n} \times \mathbf{m}$, which is in $\Delta^n \times \partial\Delta^m$.

2) If the last morphism in the path σ is the morphism $(n-1,m) \to (n,m)$ then $d_{n+m}(\sigma)$ is in the image of the poset morphism $d^n \times 1 : (\mathbf{n-1}) \times \mathbf{m} \to \mathbf{n} \times \mathbf{m}$, which is in $\Lambda^n_k \times \Delta^m$ since $k \neq n$. If the last morphism of σ is $(n, m-1) \to (n,m)$ then $d_{n+m}(\sigma)$ is in the image of the morphism $1 \times d^m : \mathbf{n} \times (\mathbf{m-1}) \to \mathbf{n} \times \mathbf{m}$, which is in $\Delta^n \times \partial\Delta^m$.

There is a pushout diagram

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & (\Delta^n \times \Delta^m)^{(T)} \arrow[d, "i"] \\
\Delta^{m+n} \arrow[r, "\sigma"'] & (\Delta^n \times \Delta^m)^{(T')}
\end{tikzcd}$$

where

$$K = \Delta^{m+n} \cap (\Delta^n \times \Delta^m)^{(T)}$$

in $\Delta^n \times \Delta^m$.

I claim that the maximal non-degenerate simplices of K have dimension $m+n-1$ so that $K = \langle S \rangle$ some set S of non-degenerate $(m+n-1)$-simplices of Δ^{m+n}.

To see this, let γ be such a maximal non-degenerate simplex of K, and write

$$(\gamma(0), \gamma'(0)) \to \cdots \to (\gamma(r), \gamma'(r))$$

for the string defining γ.

If $(\gamma(0), \gamma'(0)) \neq (0,0)$ then γ is contained in the image of one of the morphisms $d^0 \times 1$ or $1 \times d^0$, and the maximum length of non-degenerate simplices in these images in $m+n-1$. It follows that $\gamma = d_0(\sigma)$. Thus, we can assume (for otherwise γ is a face of a simplex of dimension $m+n-1$ of K) that $(\gamma(0), \gamma'(0)) = (0,0)$. We can similarly assume that $(\gamma(r), \gamma'(r)) = (m,n)$.

Suppose that the morphism

$$(\gamma(i), \gamma'(i)) \to (\gamma(i+1), \gamma'(i+1))$$

is one of the morphisms in the string defining γ. This morphism is the composite of the segment of morphisms

$$\tau_i : \ (i_1, j_1) \to \cdots \to (i_k, j_k)$$

appearing in the string defining σ. This segment is a non-degenerate simplex of $\Delta^r \times \Delta^s$ for some r, s. If this string τ_i is not minimal among all such simplices then it contains a substring of the form

$$\begin{tikzcd}
 & (i_v+1, j_v+1) \\
(i_v, j_v) \arrow[r] & (i_v+1, j_v) \arrow[u]
\end{tikzcd}$$

by Lemma 44. It follows that γ is a face of $d_{v+1}(\sigma)$, which is a face of σ as well as a face of some (unique) $\sigma' < \sigma$, so $\gamma = d_{v+1}(\sigma)$ is an $(n+m-1)$-simplex of K.

We can therefore assume that all of the strings τ_i are minimal. It follows that the string σ is minimal among strings of length $n+m$ passing through all of the points $(\gamma(i), \gamma'(i))$. It follows that γ is a face of some simplex in $(\Delta^n \times \Delta^m)^{(\emptyset)}$, and hence that γ is not interior.

If the composite

$$\mathbf{r} \xrightarrow{\gamma} \mathbf{n} \times \mathbf{m} \to \mathbf{n}$$

is not surjective, then there is an i for which $\gamma(i) + 1 < \gamma(i+1)$. This means that the corresponding segment of σ has the form

$$\begin{tikzcd}
(\gamma(i), \gamma'(i)) \arrow[r] & \dots \arrow[r] & (\gamma(i), \gamma'(i) + s) \arrow[d] & & \\
& & (\gamma(i)+1, \gamma'(i)+s) \arrow[r] & \dots \arrow[r] & (\gamma(i)+r, \gamma'(i)+s)
\end{tikzcd}$$

where $r > 1$. But then $d_v(\sigma)$ is in the image of some $d^j \times 1 : (\mathbf{n}-\mathbf{1}) \times \mathbf{m} \to \mathbf{n} \times \mathbf{m}$ (for some v determined by the point $(\gamma(i)+1, \gamma'(i)+s)$ in the string above), so that $\gamma = d_v(\sigma)$ has dimension $m+n-1$. A similar argument shows that γ has dimension $m+n-1$ if the other composite is not surjective.

With all of that in hand, suppose that σ is not maximal. Then σ contains a segment

$$\begin{tikzcd}
(i_r, j_r+1) \arrow[r] & (i_r+1, j_r+1) \\
(i_r, j_r) \arrow[u] &
\end{tikzcd}$$

and the face $d_{r+1}(\sigma)$ is not in any smaller non-degenerate $(n+m)$-simplex and is interior. There is a pushout diagram

$$\begin{tikzcd}
\langle S \rangle \arrow[r] \arrow[d] & (\Delta^n \times \Delta^m)^{(T)} \arrow[d, "i"] \\
\Delta^{n+m} \arrow[r, "\sigma"'] & (\Delta^n \times \Delta^m)^{(T')}
\end{tikzcd} \tag{13}$$

where S is a set of non-degenerate $(n+m-1)$-simplices of Δ^{n+m} that includes the simplices d^0 and d^{n+m} but is missing d^{r+1}. It follows from Lemma 8 that the morphism i is inner anodyne.

If σ is maximal, there is still a pushout diagram of the form (13), and the faces d^0 and d^{m+n} are still in the set S, but $d^k = d_k(\sigma)$ is not a member of S. In effect, $d_k(\sigma) \mapsto d^k$ under the projection $\Delta^n \times \Delta^m \to \Delta^m$ so it is not in $\Lambda^n_k \times \Delta^n$, and the composite

$$\mathbf{n} + \mathbf{m} - \mathbf{1} \xrightarrow{d_k(\sigma)} \mathbf{n} \times \mathbf{m} \to \mathbf{m}$$

is surjective, so that $d_k(\sigma)$ is not in $\Delta^n \times \partial\Delta^m$. Finally, $d_k(\sigma)$ is not a face of any smaller non-degenerate $(m+n)$-simplices because it contains the vertex $(n,0)$, so that $d_k(\sigma)$ is not a member of S. The map i is therefore inner anodyne. $\square$

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