ALGEBRAIC MODELS OF HOMOTOPY TYPES

TALK BY ANGÉLICA OSORNO; NOTES BY CHRIS KAPULKIN

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Throughout **Top** will denote a category of "nice" spaces, say, compactly generated Hausdorff spaces or simply CW-complexes.

We shall start with two preliminary definitions:

Definition 1. A map $f\colon X \longrightarrow Y$ is a *weak homotopy equivalence*, if the induced map $f_*\colon \pi_i(X,x) \longrightarrow \pi_i(Y, f(x))$ is an isomorphism for all $x \in X$ and all $i \geq 0$.

Definition 2. Two spaces are said to have the same *homotopy type*, if there exists a zig-zag of weak equivalences connecting them.

Let us point out that two spaces with isomorphic homotopy groups need not to have the same homotopy type as these isomorphisms have to be induced by a continuous map between the spaces. For example, we know that $\pi_1(X,x)$ acts on $\pi_n(X,x)$. We can easily find two spaces with the same homotopy groups but with different actions of π_1. These two spaces will not represent the same homotopy type.

An outstanding goal of algebraic topology is to find *algebraic models for homotopy types* that is algebraically defined objects, the category of which is equivalent to the homotopy category of spaces (i.e. a localization as in [GZ67] of the category **Top** at the class of weak equivalences as introduced in Definition 2).

The first attempt that we will discuss is due to Thomason [Tho80], who considered a functor:

$$\mathbf{B}\colon \mathbf{Cat} \longrightarrow \mathbf{Top}$$

given by the composite:

$$\mathbf{Cat} \xrightarrow{\mathcal{N}} \mathbf{sSets} \xrightarrow{|-|} \mathbf{Top}$$

of the *nerve functor* followed by the *geometric realization.*

Let us recall here briefly the construction of the nerve functor. Given a category $\mathcal{C}$ we define $\mathcal{NC}$ as follows:

$$\begin{array}{rcl} (\mathcal{NC})_0 & = & \text{objects of } \mathcal{C} \\ (\mathcal{NC})_1 & = & \text{morphisms of } \mathcal{C} \\ (\mathcal{NC})_2 & = & \text{composable pairs of morphisms in } \mathcal{C} \\ & \vdots & \\ (\mathcal{NC})_n & = & \text{composable } n\text{-tuples of morphisms in } \mathcal{C} \\ & \vdots & \end{array}$$

with the obvious face and degeneracy maps.

Since $\mathbf{B}$ is a functor it not only assigns to each category a space but also to each functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ a continuous map $\mathbf{B}F\colon \mathbf{B}\mathcal{C} \longrightarrow \mathbf{B}\mathcal{D}$. This can be extended to natural transformations as follows. First, let us observe that there is a correspondence:

$$\left\{ \begin{array}{c} \text{natural transformations} \\ \varphi\colon F \Longrightarrow G \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{functors } H\colon \mathcal{C} \times \mathcal{I} \longrightarrow \mathcal{D} \text{ s.th.} \\ H(-,0) = F \text{ and } H(-,1) = G \end{array} \right\},$$

where $\mathcal{I}$ denotes a category with two objects: 0 and 1, and one non-identity arrow $0 \longrightarrow 1$. It is easy to see that $\mathbf{B}\mathcal{I} \cong [0,1]$ the unit interval. Moreover, since both the nerve functor and the geometric realization preserve binary products, we have:

$$\mathbf{B}(\mathcal{C} \times \mathcal{I}) \cong \mathbf{B}\mathcal{C} \times \mathbf{B}\mathcal{I} \cong \mathbf{B}\mathcal{C} \times [0,1].$$

Thus we may conclude that $\mathbf{B}$ takes a natural transformation $\varphi\colon F \Longrightarrow G$ to a homotopy from $\mathbf{B}F$ to $\mathbf{B}G$. This discussion can be summarized within the following theorem:

Theorem 3 (Thomason)**.** *The functor* $\mathbf{B}\colon \mathbf{Cat} \longrightarrow \mathbf{Top}$ *induces an equivalence of homotopy categories, where the weak equivalences in* $\mathbf{Cat}$ *are functors* $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ *such that* $\mathbf{B}F$ *is a weak equivalence in* $\mathbf{Top}$*.*

This theorem gives a way of thinking about homotopy types as categories. It has however two main disadvantages:

(1) Weak equivalences of categories as defined above are hard to understand and work with. In fact, if we want to check whether a functor is a weak equivalence, we still need to check in the category $\mathbf{Top}$.
(2) The "inverse" functor of $\mathbf{B}$ is hard to understand and work with.

The difficulties arising in Thomason's Theorem 3 suggest that our original goal may be hard to achieve. In the remaining part of the notes we will address this question for spaces that have only finitely many non-trivial homotopy groups. As before, we begin by introducing the basic notions:

Definition 4. A map $f\colon X \longrightarrow Y$ is an $(n+1)$-*equivalence*, if for all $x \in X$ the induced map $f_*\colon \pi_i(X,x) \longrightarrow \pi_i(Y, f(x))$ is an isomorphism for $i \leq n$ and is surjective for $i = n+1$.

Definition 5. Two spaces have the same n-*type*, if there exists a zig-zag of $(n+1)$-equivalences between them.

When studying spaces up to $(n+1)$-equivalence, one may want to restrict attention to spaces whose ith homotopy groups vanish for $i > n$. This is justified by the following simple observation: by adding cells to a given space X we may build a new space X' together with an $(n+1)$-equivalence $f\colon X \longrightarrow X'$ such that $\pi_i(X', x') = 0$ for $i > n$. Let therefore $\mathbf{Top}^{\leq n}$ denote the full subcategory of $\mathbf{Top}$ consisting of spaces X such that $\pi_i(X, x) = 0$ for all $x \in X$ and $i > n$. We will try to describe the category $\mathrm{Ho}(\mathbf{Top}^{\leq n})$ for $n = 1, 2$.

Before dealing with a general case, let us start by understanding the subcategories $\mathrm{Ho}(\mathbf{Top}_*^{\leq n})$ of connected n-types for $n = 1, 2$.

If $n = 1$, then a space $X \in \mathbf{Top}_*^{\leq 1}$ is determined (up to a weak equivalence) by its fundamental group $\pi_1(X, x)$. Given an arbitrary group G we can regard it as a category with one object and form $\mathbf{B}G$. Moreover, there is an isomorphism:

$$[\mathbf{B}G, \mathbf{B}H] \cong \mathrm{Hom}(G, H),$$

where $[X, Y]$ denotes the set of homotopy classes of maps from X to Y; and hence $\mathbf{B}$ exhibits an equivalence of categories:

$$\mathrm{Ho}(\mathbf{Top}_*^{\leq 1}) \simeq \mathbf{Grp}.$$

When $n = 2$ a space is not determined by providing a group $\pi_1(X, x)$ and an abelian group $\pi_2(X, x)$. One needs to equip $\pi_2(X, x)$ with an action of $\pi_1(X, x)$. This is captured exactly by the notion of a crossed module.

Definition 6. A *crossed module* M consists of a homomorphism $\mathrm{d}\colon H \longrightarrow G$ of groups together with an action of G on H (denoted by $\cdot$) satisfying the following conditions:

1. $\mathrm{d}(g \cdot h) = g\mathrm{d}(h)g^{-1}$ (*equivariance*),
2. $\mathrm{d}(h_1) \cdot h_2 = h_1 h_2 h_1^{-1}$.

Below we give a few simple examples of crossed modules:

Examples 7.
1. If N is a normal subgroup of G, then the inclusion $N \hookrightarrow G$ together with the action given by conjugation is a crossed module.
2. The inclusion $H \hookrightarrow \mathrm{Aut}(H)$ via inner automorphisms is a crossed module.
3. The connecting homomorphism $\partial\colon \pi_2(X, A, x) \longrightarrow \pi_1(A, x)$ in the long exact sequence of a pair is a crossed module.
4. Given a fibration $F \hookrightarrow E \longrightarrow B$, the induced map $\pi_1(F) \longrightarrow \pi_1(E)$ is a crossed module.

There is also a natural notion of a crossed module homomorphism:

Definition 8. Let $M = (\mathrm{d}\colon H \longrightarrow G)$ and $M' = (\mathrm{d}'\colon H' \longrightarrow G')$ be two crossed modules. A *morphism of crossed modules* $M \longrightarrow M'$ is a pair of group homomorphisms $f_1\colon H \longrightarrow H'$ and $f_2\colon G \longrightarrow G'$ such that

$$\mathrm{d}' \circ f_1 = f_2 \circ \mathrm{d} \qquad \text{and} \qquad f_1(g \cdot h) = f_2(g) \cdot f_1(h).$$

The category of crossed modules (and their morphisms) will be denoted by **CrMod**. We may now state our main theorem regarding connected 2-types.

Theorem 9. *There is a functor* $\mathbf{B}\colon \mathbf{CrMod} \longrightarrow \mathbf{Top}$*, with* $\mathbf{B}M$ *connected for any* M*, and exhibiting an equivalence of categories* $\mathbf{CrMod} \simeq \mathrm{Ho}(\mathbf{Top}_*^{\leq 2})$*.*

Let us make a few comments about the above theorem. First, we shall notice that the homotopy groups of $\mathbf{B}M$ are fully determined by the map d as follows:

$$\pi_i(\mathbf{B}M) = \begin{cases} \mathrm{Coker}(\mathrm{d}) & \text{if } i = 1 \\ \mathrm{Ker}(\mathrm{d}) & \text{if } i = 2 \\ 0 & \text{for } i > 2. \end{cases}$$

Conversely, a crossed module is uniquely determined by the following data:

- $\pi_1 := \pi_1(\mathbf{B}M)$,
- $\pi_2 := \pi_2(\mathbf{B}M)$,
- and the *k-invariant* i.e. a cohomology class $[c] \in H^3(\pi_1; \pi_2)$.

If we wish to generalize this equivalence to higher n's we can proceed in two ways. We might find a higher-dimensional refinement of the notion of a crossed module or replace crossed modules with an equivalent notion that will be easier to generalize. In the former case, we arrive at the notion of a *crossed complex* that we will not discuss here. In the latter we shall recall/introduce the notion of a 2-group:

Definition 10. A 2*-group* G is a monoidal groupoid such that all objects in G are invertible up to isomorphism.

Denoting the category of 2-groups by 2-**Grp** we obtain the following proposition explaining in what sense 2-groups can serve as a replacement for crossed modules:

Proposition 11. *There is an equivalence of categories* **CrMod** $\simeq$ 2-**Grp***.*

As a final remark (for the connected case) let us mention that following this idea Loday [Lod82] proposed the **Cat**-n-groups that classify homotopy n-types in a similar way as above.

After dealing with the connected case, we may now try to generalize our results to all, not necessarily connected, spaces. That in fact requires passing from groups and 2-groups to groupoids and 2-groupoids.

Traditionally, we begin with the case $n = 1$, where we have a pair of functors:

$$\mathbf{B}\colon \mathbf{Gpd} \rightleftarrows \mathbf{Top} : \Pi_1,$$

where $\Pi_1 X$ is the *fundamental groupoid* of X i.e. a groupoid whose objects are points of X and given any $x, y \in X$ we define $\mathrm{Hom}(x, y)$ as the set of homotopy classes of paths from x to y.

We shall note here that given any groupoid $\mathcal{G}$ the space $\mathbf{B}\mathcal{G}$ is a 1-type. Moreover, we can define the first two homotopy groups of a groupoid $\mathcal{G}$ by:

$$\pi_0\mathcal{G} := \mathrm{ob}\,\mathcal{G}/\cong \qquad \text{and} \qquad \pi_1(\mathcal{G}, x) := \mathrm{Hom}_{\mathcal{G}}(x, x).$$

It is easy to see that these are preserved by the functor $\mathbf{B}$. Conversely, the homotopy groups of a homotopy 1-type are preserved by the functor Π_1. This consideration can be summarized within the following theorem:

Theorem 12. *The functors* $\mathbf{B}$ *and* Π_1 *exhibit an equivalence of categories* $\mathrm{Ho}(\mathbf{Gpd}) \simeq \mathrm{Ho}(\mathbf{Top}^{\leq 1})$.

Note that even though this statement may be similar to Thomason's Theorem 3, it actually gives a nice description of the homotopy 1-types. Indeed, under passage to $\mathrm{Ho}(\mathbf{Gpd})$ we invert only categorical equivalences (i.e. fully faithful and essentially surjective functors) that are easy to describe without mentioning weak homotopy equivalences. Also, in this case the "inverse" functor of $\mathbf{B}$ has a nice description (as we have just seen).

Now let us consider $n = 2$. We can exhibit a similar equivalence between homotopy 2-types and 2-groupoids. Let us recall the definition of a 2-groupoid and state the theorem.

Definition 13. A 2-*groupoid* is a 2-category such that all the 2-cells are invertible and all the 1-cell are weakly invertible (up to a 2-cell).

Theorem 14. *There is a pair of functors* $\mathbf{B}\colon 2\text{-}\mathbf{Gpd} \rightleftarrows \mathbf{Top} : \Pi_2$ *exhibiting an equivalence of homotopy categories:*

$$\mathrm{Ho}(2\text{-}\mathbf{Gpd}) \simeq \mathrm{Ho}(\mathbf{Top}^{\leq 2})$$

Since 1-types are classified by (1-)groupoids and 2-types by 2-groupoids, it is natural to expect that the n-types will be classified by n-groupoids. This statement is known as Grothendieck's Homotopy Hypothesis.

Conjecture 15 (Homotopy Hypothesis, Grothendieck)**.** *Weak n-groupoids model homotopy n-types.*

References

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