FIBER BUNDLES AND UNIVALENCE

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This talk presents a proof that a universal Kan fibration is univalent. The talk was given by Ieke Moerdijk during the conference *Mathematics: Algorithms and Proofs 2011*, and these notes prepared by Chris Kapulkin (who is the only person responsible for any mistakes that can be found in them).

These notes are organized in three parts. The first part is an introduction to the theory of topological bundles that will be used in our proof. In the second part we gather the necessary facts about the category of simplicial sets. The last part contains the proof that the universal Kan fibration is univalent.

Let us point here out that these notes should not be read like a research paper as in several places they lack a necessary explanation. This is due to the fact that they are notes from a 30 minute talk.

1. Fiber bundles

In this section we review some basic facts about topological bundles following [Ste51]. This theory works in essentially the same way for simplicial sets (see, for example, [May67]). We recall it here for topological spaces to make the presentation clearer.

1.1. **Fiber bundles.**

We begin by recalling the notion of a fiber bundle.

Definition 1. A surjective map $Y \to X$ is called a *fiber bundle with fiber* F, if for some open cover $X = \bigcup\limits_{i \in I} U_i$ the square:

$$\begin{array}{ccc} U_i \times F & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ U_i & \hookrightarrow & X \end{array}$$

is a pullback square for every $i \in I$, where the left vertical arrow is the first projection.

Example 2. Let M be an n-dimensional smooth real manifold. The fiber of the tangent bundle of M is isomorphic to $\mathbb{R}^n$.

Let $Y \to X$ be a fiber bundle with fiber F and let $X = \bigcup_{i \in I} U_i$ be a cover of X as in the definition 1. Let U_i, U_j be two sets in this cover. Taking pullbacks along their inclusions into X and restricting to $U_{ij} = U_i \cap U_j$ gives an automorphism:

$$\begin{array}{ccc} U_{ij} \times F & \xrightarrow{\phi_{ij}} & U_{ij} \times F \\ & \searrow \quad \swarrow & \\ & U_{ij} & \end{array}$$

fibered over U_{ij} and for each $u \in U_{ij}$ an automorphism $\phi_{ij}(u)\colon F \to F$ i.e. a map (a cocycle) $U_{ij} \to \mathrm{Aut}(F)$. That leads to the following definition:

Definition 3. Let $G \subseteq \mathrm{Aut}(F)$ be a group of homeomorphisms of F. A *fiber bundle with a structure group* G (or, shortly, a G-*bundle*) is a fiber bundle with fiber F such that the collection of maps $\{U_i \cap U_j = U_{ij} \to \mathrm{Aut}(F)\}_{i,j \in I}$ factors through $G \subseteq \mathrm{Aut}(F)$.

Example 4. A vector bundle is a fiber bundle whose fiber is a vector space V and whose structure group is a subgroup of $\mathrm{GL}(V)$.

1.2. **Principal G-bundles.** We next recall the notion of a principal bundle.

Definition 5. Let G be a group. A *principal G-bundle* is a surjective map $P \to X$ equipped with a free and transitive G-action on the fibers of P such that there exists an open cover $X = \bigcup_{i \in I} U_i$ and a family of G-equivariant maps $\{U_i \times G \hookrightarrow P\}_{i \in I}$ such that all the squares of the form:

$$\begin{array}{ccc} U_i \times G & \hookrightarrow & P \\ \downarrow & & \downarrow \\ U_i & \hookrightarrow & X \end{array}$$

are pullbacks, where the left vertical arrow is the first projection.

1.3. **Correspondence between fiber bundles and principal G-bundles.** Let G be a group. Given a principal G-bundle, we can form a fiber bundle with structure group G by:

$$\begin{array}{ccc} P & & P \times_G F \\ \downarrow & \longmapsto & \downarrow \\ X & & X \end{array} \tag{1}$$

where $P \times_G F$ is a a quotient of the space $P \times F$ by an equivalence relation generated by $(p, gy) \sim (pg, y)$.

Theorem 6. *The assignation (1) defines a bijection between principal G-bundles and fiber bundles with structure group G.*

We leave this theorem without a proof. However, we wish to point out that if $Y \to X$ is a fiber bundle with fiber F and $G = \mathrm{Aut}(F)$, then the correspondence above turns the bundle $Y \to X$ into:

$$\begin{array}{c} \mathrm{Fr}(F, Y) \\ \big\downarrow \\ X \end{array}$$

whose fiber over $x \in X$ is $\mathrm{Iso}(F, Y_x)$.

The notation $\mathrm{Fr}(F, Y)$, coming originally from framed bundles, is justified by the example below.

Example 7. Let $Y \to X$ be a vector bundle with fiber V. An associated principal bundle is a *framed bundle* $\mathrm{Fr}(V, Y) \to X$. A point (or a *frame*) over $x \in X$ in $\mathrm{Fr}(V, Y) \to X$ is a linear isomorphism $\mathbb{R}^n \to Y_x$ i.e. a choice of basis in the fiber Y_x.

1.4. **Classifying space of principal G-bundles.** We finish our background on topological bundles by recalling the construction of the classifying space of principal G-bundles.

Definition 8. Let G be a group. The *classifying space* of principal G-bundles is a space $\mathbf{B}G$ together with a principal G-bundle

$$\begin{array}{c} \mathbf{E}G \\ {\scriptstyle \pi}\big\downarrow \\ \mathbf{B}G \end{array}$$

with a property that every principal G-bundle can be obtained as a pullback from it i.e. the following assignment is an isomorphism:

$$[X, \mathbf{B}G] \xrightarrow{\quad (-)^*\pi \quad} \left\{ \begin{array}{c} \text{Iso-classes of principal} \\ G\text{-bundles over } X \end{array} \right\}$$

Remark 9. Such a space is uniquely determined (up to homotopy equivalence) by the following two properties:

(1) $\mathbf{E}G$ is a space equipped with a free action of G,
(2) $\mathbf{E}G$ is contractible.

Note that by condition (1) we can put $\mathbf{B}G = \mathbf{E}G/G$.

As a final remark in this section we point out that by the long exact homotopy sequence of fibration, we obtain a homotopy equivalence:

$$G \simeq \Omega(\mathbf{B}G, *). \tag{2}$$

2. Background on simplicial sets

In this section we gather some facts about the category **sSets** of simplicial sets with a special emphasis on the notion of the minimal fibration.

Definition 10. Let $E \to X$ be a Kan fibration. We can form a fibration:

$$\begin{array}{c} \mathrm{Eq}(E) \\ \downarrow \\ X \times X \end{array}$$

whose n-simplices are of the form: (x, y, p), where $x, y \colon \Delta[n] \to X$ are n-simplices of X and $p \colon x^*E \to y^*E$ is a weak equivalence.

Remark 11. Every fibered weak equivalence:

$$\begin{array}{ccc} E & \xrightarrow{\simeq} & E' \\ \downarrow & \swarrow & \\ X & & \end{array}$$

induces a fibered weak equivalence:

$$\begin{array}{ccc} \mathrm{Eq}(E) & \xrightarrow{\simeq} & \mathrm{Eq}(E') \\ \downarrow & \swarrow & \\ X \times X & & \end{array}$$

We now move towards minimal fibrations (cf. [Qui68, May67]). Recall that for any Kan fibration $E \to X$ there exists a minimal fibration $M \to X$ equivalent to it:

$$\begin{array}{ccc} E & \xrightarrow{\simeq} & M \\ \downarrow & \swarrow & \\ X & & \end{array}$$

Let us observe that since an equivalence between two minimal Kan fibration is an isomorphism, we have the following:

Proposition 12. *Let $M \to X$ be a minimal fibration. Then there is an equivalence:*

$$\begin{array}{ccc} \mathrm{Eq}(M) & \xrightarrow{\simeq} & \mathrm{Iso}(M) \\ \downarrow & \swarrow & \\ X \times X & & \end{array}$$

where $\mathrm{Iso}(M)_{(x,y)} = \mathrm{Iso}(M_x, M_y)$.

3. The Univalence Axiom in the category of simplicial sets

Let $\pi\colon E \to B$ be a universal Kan fibration in **sSets** i.e. any fibration (with some size restriction) can be obtained in a unique (up to homotopy) way as a pullback from $\pi\colon E \to B$.

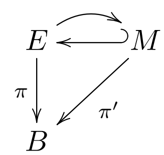
Our goal in this section is to prove the following theorem.

Theorem 13 (Univalence Axiom)**.** *Let $\pi\colon E \to B$ be a universal fibration. Then the obvious map*

$$\mathrm{Path}(B) \to \mathrm{Eq}(E),$$

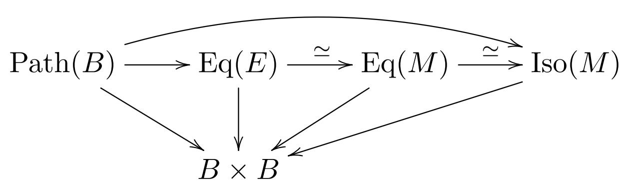
where $\mathrm{Path}(B) = B^{\Delta[1]}$*, is a weak equivalence.*

Proof. Let $\pi'\colon M \to B$ be a minimal fibration contained in $\pi\colon E \to B$ i.e.



Then the fibration $\pi'\colon M \to B$ is universal for fiber bundles.

First, let us observe that it suffices to prove that the map $\mathrm{Path}(B) \to \mathrm{Iso}(M)$ in the diagram below is a weak equivalence:

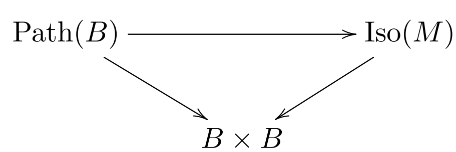


Take $b_0 \in B$ and let $F_0 = M_{b_0}$ be a fiber over b_0 (note that $F_0 \simeq E_{b_0}$). If $B_0 \subseteq B$ is the connected component of b_0, then an obvious map $M_0 = \pi'^{-1}(B_0) \to B_0$ is a minimal and universal F_0-bundle.

By the correspondence between fiber bundles and principal bundles the obvious map $\mathrm{Fr}(F_0, M_0) \to B_0$ is a universal principal $\mathrm{Aut}(F_0)$-bundle i.e. $B_0 = \mathbf{B}\mathrm{Aut}(F_0)$, so by (2) we obtain:

$$\Omega(B_0, b_0) \simeq \mathrm{Aut}(F_0). \tag{3}$$

By pulling back the diagram:



along the inclusion $\{b_0\} \times B_0 \hookrightarrow B \times B$ we obtain:

$$\begin{array}{ccc} \mathrm{P}(B_0, b_0) & \longrightarrow & \mathrm{Fr}(F_0, M_0) \\ & \searrow \quad \swarrow & \\ & B_0 & \end{array}$$

where $\mathrm{P}(B_0, b_0)$ denotes the pointed path space of B_0. Recall, that for a map between fibrations over a connected base to be an equivalence, it is sufficient that the induced map between the fibers over just one base point be an equivalence. By (3) the fibers over $b_0 \in B_0$ of both fibrations in the diagram above are equivalent, so the horizontal map in this diagram is an equivalence.

We finish the proof by observing that we can repeat the above argument for each connected component of B separately exhibiting the fibered equivalence:

$$\begin{array}{ccc} \mathrm{Path}(B) & \xrightarrow{\simeq} & \mathrm{Iso}(M) \\ & \searrow \quad \swarrow & \\ & B \times B & \end{array}$$

as desired. □

We leave the following question as an open problem.

Question 14. One can observe that being univalent is a property of a fibration in a simplicial model category. We can therefore ask whether there exist other model categories whose universal fibrations satisfy the Univalence Axiom.

References

[May67] J. Peter May, *Simplicial objects in algebraic topology*, Mathematical Studies, vol. 11, Van Nostrand, 1967.

[Qui68] Daniel Quillen, *The geometric realization of a Kan fibration is a Serre fibration*, Proceedings of the American Mathematical Society **19** (1968), 1499–1500.

[Ste51] Norman Earl Steenrod, *The topology of fibre bundles*, Princeton University Press, 1951.