INTRODUCTION TO QUILLEN MODEL CATEGORIES

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This talk is meant to be an introduction to the theory of model categories for people who are not familiar with algebraic topology or (abstract) homotopy theory. The talk was given by Ieke Moerdijk during the conference *Mathematics: Algorithms and Proofs 2011*, and these notes prepared by Chris Kapulkin (who is the only person responsible for any mistakes that can be found in them).

The notes are organized in two parts. In the first part we introduce model categories, give a few examples, and present some basic notions. In the second part we focus on the category **sSets** of simplicial sets that carries a model structure that enjoys several good properties.

We recommend Mark Hovey's book [Hov99] for further reading.

1. Definition and first notions

In this section we introduce basic notions from the theory of model categories. Before defining a model category, we need the notion of a retract.

Definition 1. A map $g\colon A \to B$ is said to be a *retract* of $f\colon X \to Y$, if there exists a commutative diagram:

$$
\begin{array}{ccccc}
A & \overset{i}{\hookrightarrow} & X & \overset{r}{\longrightarrow} & A \\
g\downarrow & & f\downarrow & & \downarrow g \\
B & \underset{j}{\hookrightarrow} & Y & \underset{s}{\longrightarrow} & B
\end{array}
$$

with $ri = 1_A$ and $sj = 1_B$.

Now we are ready to introduce model categories. We will follow the definition given in an excellent article by Daniel Quillen [Qui69].

Definition 2. Let $\mathcal{E}$ be a category. A *Quillen model structure* on $\mathcal{E}$ is given by 3 classes of maps: fibrations, cofibrations, and weak equivalences satisfying axioms CM1–CM5:

CM1. $\mathcal{E}$ has all finite limits and colimits.
CM2. (2-out-of-3) Let f, g be a composable pair of morphisms in $\mathcal{E}$. If any two of f, g, $g \circ f$ are weak equivalences, so is the third.
CM3. All three classes are closed under retracts.

CM4. Given a commutative square of the form:

$$
\begin{array}{ccc}
A & \xrightarrow{\ a\ } & X \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle f} \\
B & \xrightarrow[\ b\ ]{} & Y
\end{array}
$$

where c is a cofibration and f is a fibration, if moreover either c or f is a weak equivalence, then there exists a diagonal filler d making both triangles in the diagram below commute:

$$
\begin{array}{ccc}
A & \xrightarrow{\ a\ } & X \\
\downarrow{\scriptstyle c} & \overset{d}{\nearrow} & \downarrow{\scriptstyle f} \\
B & \xrightarrow[\ b\ ]{} & Y
\end{array}
$$

CM5. Any map f can be written as $f = pi$ where p is a fibration and i is a cofibration in two ways: one in which p is also a weak equivalence and one in which i is also a weak equivalence.

In many cases such as the categories of topological spaces or chain complexes these axioms are very powerful, but rather hard to verify. Below we give a few examples when it is easy to verify them, but at the same time they give very little information about the structure of a category.

Of course, if $\mathcal{E}$ is any category having all finite limits and colimits, $\mathcal{E}$ can be trivially equipped with a Quillen model category structure by taking any two classes to be all maps of $\mathcal{E}$ and the third class to be all isomorphisms in $\mathcal{E}$. Less obvious, yet still easy examples are the following.

Examples 3.
- The category **Gpd** of small groupoids can be equipped with a Quillen model category structure, where weak equivalences are categorical equivalences and fibrations are the Grothendieck fibrations. We urge the Reader to characterize the class of cofibrations as an exercise.
- The category **Cat** of small categories can be equipped with a Quillen model category structure in a similar way as above. The class of weak equivalences is again the class of categorical equivalences and fibrations are the isofibrations (recall that $F\colon \mathcal{E} \to \mathcal{F}$ is an *isofibration*, if it has a lifting property with respect to all isomorphisms in $\mathcal{F}$).

Remark 4. Any two classes from the definition of a Quillen model structure determine the third.

Definition 5. Let $\mathcal{E}$ be a Quillen model category. An object $X \in \mathcal{E}$ is called *fibrant* (resp. *cofibrant*), if the unique map $X \to \mathbf{1}$ (resp. $\mathbf{0} \to X$) is a fibration (resp. cofibration).

Remark 6. A model structure on a category is completely determined by its cofibrations and its fibrant objects or, dually, by its fibrations and its cofibrant objects.

Let us point here out that the original goal of Quillen was to describe the *homotopy category* $\text{Ho}(\mathcal{E}) := \mathcal{E}[\text{w.e.}^{-1}]$ of $\mathcal{E}$ obtained by formally inverting all the weak equivalences in $\mathcal{E}$ and compare $\text{Ho}(\mathcal{E})$ for different $\mathcal{E}$'s.

Definition 7. Let $\mathcal{E}$, $\mathcal{F}$ be two model categories. An adjoint pair $F\colon \mathcal{E} \rightleftarrows \mathcal{F} : G$ is called a *Quillen pair*, if F preserves cofibrations and G preserves fibrations.

Remark 8. Calling (co)fibration that is also a weak equivalence a *trivial (co)fibration* we can say that $F\colon \mathcal{E} \rightleftarrows \mathcal{F} : G$ is a Quillen pair if F preserves cofibrations and trivial cofibrations (or equivalently, G preserves fibrations and trivial fibrations).

If $F\colon \mathcal{E} \rightleftarrows \mathcal{F} : G$ is a Quillen pair, it induces a well-defined adjoint pair:

$$\text{Ho}(\mathcal{E}) \underset{RG}{\overset{LF}{\rightleftarrows}} \perp \text{Ho}(\mathcal{F})$$

Definition 9. A *Quillen equivalence* is a Quillen pair with the additional property that $LF \dashv RG$ is an equivalence of categories.

Remark 10. There are many ways to define what a Quillen equivalence is. For example, one can say that for any cofibrant $X \in \mathcal{E}$ and any fibrant $Y \in \mathcal{F}$: a map $X \xrightarrow{\alpha} GY$ is a weak equivalence if and only if its adjoint transpose $FX \xrightarrow{\widehat{\alpha}} Y$ is a weak equivalence.

Definition 11. A model category $\mathcal{E}$ is said to be *left* (resp. *right*) *proper*, if the pushout (resp. pullback) of a weak equivalence along a cofibration (resp. fibration) is again a weak equivalence. $\mathcal{E}$ is *proper*, if it is both left and right proper.

Definition 12. A model category $\mathcal{E}$ is said to be *cartesian*, if for any two cofibrations $A \to B$ and $X \to Y$, the map

$$A \times Y \cup_{A \times X} B \times X \to B \times Y$$

is again a cofibration, trivial, if one of $A \to B$ and $X \to Y$ is. Here $A \times Y \cup_{A \times X} B \times X$ is the pushout:

$$
\begin{array}{ccc}
A \times X & \longrightarrow & A \times Y \\
\downarrow & & \downarrow \\
B \times X & \longrightarrow & \bullet
\end{array}
$$

2. The category of simplicial sets

In this section we introduce the category **sSets** of simplicial sets that carries a nice model structure (which, for example, is proper and cartesian).

We begin by recalling that the category **sSets** of simplicial sets is the category of presheaves on the category Δ which is the category of finite ordinals $[n] = \{0, 1, \ldots, n\}$ and $\leq$-preserving maps between them.

Let $\Delta[n] := \Delta(-, [n])$ be the representable. Geometrically, one should think of $\Delta[n]$ as a standard n-simplex.

We next list a couple of examples of subobjects of $\Delta[n]$.

Examples 13. The following are subobjects of $\Delta[n]$:

- $\Delta[n-1] \overset{\partial^i}{\hookrightarrow} \Delta[n]$ is the image under the Yoneda embedding of $\partial_i \colon [n-1] \to [n]$ which skips i with $0 \leq i \leq n$.
- $\partial \Delta[n] = \bigcup\limits_{i=0}^{n} \Delta[n-1] \hookrightarrow \Delta[n]$ is the *boundary* of $\Delta[n]$.
- $\Lambda^k[n] = \bigcup\limits_{i \neq k} \Delta[n-1] \hookrightarrow \Delta[n]$ is the *k-th horn*.

Definition 14. A *Kan fibration* is a map $f \colon Y \to X$ of simplicial sets for which every commutative square of the form:

$$
\begin{array}{ccc}
\Lambda^k[n] & \longrightarrow & Y \\
\downarrow & \nearrow & \downarrow f \\
\Delta[n] & \longrightarrow & X
\end{array}
$$

admits a diagonal filler as indicated.

Definition 15. A simplicial set Y is called a *Kan complex*, if the unique map $Y \to \mathbf{1}$ is a Kan fibration.

We can now define a model structure on the category **sSets**: its cofibrations are the (pointwise) monomorphisms and its fibrant objects are precisely the Kan complexes.

Here are some basic properties of (Kan) fibrations:

Proposition 16.   *(1) The class of fibrations is closed under composition, pullbacks along any map and Π-functors along fibrations.*
(2) If both $Y \to X$ and $Z \to X$ are fibrations, then so is $Y + Z \to X$.
(3) The initial algebra for a W-type $\mathrm{W}(f)$ given by a fibration $f \colon B \to A$ of simplicial sets is a Kan complex.

REFERENCES

[Hov99] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, Rhode Island, 1999.

[Qui69] Daniel Quillen, *Rational homotopy theory*, The Annals of Mathematics **90** (1969), no. 2, 205–295.