THE HOMOTOPY THEORY OF TYPE THEORIES

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Abstract. We construct a left semi-model structure on the category of intensional type theories (precisely, on $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$). This presents an ∞-category of such type theories; we show moreover that there is an ∞-functor Cl_∞ from there to the ∞-category of suitably structured quasi-categories.

This allows a precise formulation of the conjectures that intensional type theory gives internal languages for higher categories, and provides a framework and toolbox for further progress on these conjectures.

1. Introduction

Homotopy Type Theory (HoTT) has often been described as the **internal language of ∞-categories**. Informally, this covers a range of ideas: theorems and constructions given in HoTT can be interpreted in a variety of higher-categorical settings, and conversely, many higher-categorical notions can be translated into type theory. Various results in these directions have been given: some proven, some conjectured, some only informally sketched (see, for instance, [Joy11, Shu12, Shu15b, Shu15a, Shu15c, Kap15]).

By analogy with established "internal languages" in 1-categorical settings, one hopes for a single master statement subsuming all of these: the existence of suitable **equivalences** between some (higher) categories of type theories and ∞-categories.

The first contribution of the current paper is a framework for precise statements of such conjectures. We do so by assembling type theories into a higher category, and giving a functor Cl_∞ from this to a higher category of suitably structured quasicategories. The internal language conjecture then states: Cl_∞ is an equivalence of higher categories.

The other main contribution is a left semi-model structure on the category of type theories. This gives a tractable and explicit presentation of the higher category thereof, which we hope will provide a solid base for further progress on the conjectures.

In a little more detail: we work with "type theories" as contextual categories or categories with attributes (CwA's), keeping our results independent of the correspondence between these and syntactically presented theories. We assume Id-, Σ-, and unit types throughout; we consider also the extension to Π-types.

Two technical tools of the paper may be of independent interest. One (small, but useful and to our knowledge new) is a notion of equivalence between arbitrary objects of a CwA. The other is the construction of the CwA of span-equivalences in a given CwA, a powerful tool for constructing equivalences between CwA's.

In Section 5, we make use of some results from our forthcoming article [KL16], currently in preparation. However, those results may be treated as black boxes; the present paper can be read as essentially self-contained.

During the preparation of this paper, we learned that Valery Isaev has independently given a similar construction in [Isa16], defining a (full) model structure on a slightly different category of type theories (assuming an interval type, instead of Martin-Löf identity types).

2. Background

In this section, we review the necessary background on categorical models of type theory. We recall the definition of a contextual category and introduce the notation for working with

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them. We then investigate their homotopy-theoretic properties, employing the language of fibration categories.

Contextual categories and functors

We choose to work with contextual categories as our model of type theory. These were introduced by Cartmell in his thesis [Car78] and studied by Streicher [Str91] and more recently in a series of papers by Voevodsky (see e.g., [Voe15a, Voe16, Voe15c, Voe15b]).

Definition 2.1. A **contextual category** $\mathbf{C}$ consists of the following data:

(1) a category $\mathbf{C}$;
(2) a grading of objects as $\mathrm{Ob}\,\mathbf{C} = \coprod_{n:\mathbb{N}} \mathrm{Ob}_n\,\mathbf{C}$;
(3) an object $1 \in \mathrm{Ob}_0\,\mathbf{C}$;
(4) **father** operations $\mathrm{ft}_n : \mathrm{Ob}_{n+1}\,\mathbf{C} \longrightarrow \mathrm{Ob}_n\,\mathbf{C}$ (whose subscripts we suppress);
(5) for each $\Gamma \in \mathrm{Ob}_{n+1}\,\mathbf{C}$, a map $p_\Gamma : \Gamma \longrightarrow \mathrm{ft}\,\Gamma$ (the **canonical projection** from Γ, distinguished in diagrams as $\longrightarrow\!\!\triangleright$);
(6) for each $\Gamma \in \mathrm{Ob}_{n+1}\,\mathbf{C}$ and $f : \Delta \longrightarrow \mathrm{ft}\,\Gamma$, an object $f^*\Gamma$ together with a **connecting map** $f.\Gamma : f^*\Gamma \longrightarrow \Gamma$;

such that:

(7) 1 is the unique object in $\mathrm{Ob}_0\,\mathbf{C}$;
(8) 1 is a terminal object in $\mathbf{C}$;
(9) for each $\Gamma \in \mathrm{Ob}_{n+1}\,\mathbf{C}$, and $f : \Delta \longrightarrow \mathrm{ft}\,\Gamma$, we have $\mathrm{ft}(f^*\Gamma) = \Delta$, and the square

$$\begin{array}{ccc} f^*\Gamma & \xrightarrow{f.\Gamma} & \Gamma \\ {\scriptstyle p_{f^*\Gamma}}\downarrow & \lrcorner & \downarrow{\scriptstyle p_\Gamma} \\ \Delta & \xrightarrow{f} & \mathrm{ft}\,\Gamma \end{array}$$

is a pullback (the **canonical pullback** of Γ along f); and
(10) these canonical pullbacks are strictly functorial: that is, for $\Gamma \in \mathrm{Ob}_{n+1}\,\mathbf{C}$, $\mathrm{id}^*_{\mathrm{ft}\,\Gamma}\Gamma = \Gamma$ and $\mathrm{id}_{\mathrm{ft}\,\Gamma}.\Gamma = \mathrm{id}_\Gamma$; and for $\Gamma \in \mathrm{Ob}_{n+1}\,\mathbf{C}$, $f : \Delta \longrightarrow \mathrm{ft}\,\Gamma$ and $g : \Theta \longrightarrow \Delta$, we have $(fg)^*\Gamma = g^*f^*\Gamma$ and $fg.\Gamma = f.\Gamma \circ g.f^*\Gamma$.

Contextual categories can be easily seen as models of an essentially algebraic theory with sorts indexed by $\mathbb{N} + \mathbb{N} \times \mathbb{N}$. As such, they come with a canonical notion of morphism: a **contextual functor** $F : \mathbf{C} \longrightarrow \mathbf{D}$ between contextual categories is a homomorphism between them, regarded as models of an essentially algebraic theory. Explicitly, F is a functor preserving on the nose all the structure of Definition 2.1: the grading on objects, the terminal object, the father maps, the dependent projections, the canonical pullbacks, and the connecting maps.

We denote the category of contextual categories and contextual functors by CxlCat.

Notation 2.2. Given $\Gamma \in \mathrm{Ob}_n\,\mathbf{C}$, we write $\mathrm{Ty}_{\mathbf{C}}(\Gamma)$ for the set of objects $\Gamma' \in \mathrm{Ob}_{n+1}\,\mathbf{C}$ such that $\mathrm{ft}(\Gamma') = \Gamma$, and call these **types in context** Γ. For $A \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$, we write $\Gamma.A$ for A considered as an object of $\mathbf{C}$, p_A for the projection $p_{\Gamma.A} : \Gamma.A \longrightarrow \Gamma$, and $f.A : f^*(\Gamma.A) \longrightarrow \Gamma.A$ for the connecting map $f.(\Gamma.A)$. For each $f : \Gamma' \longrightarrow \Gamma$, we have a map $f^* : \mathrm{Ty}_{\mathbf{C}}(\Gamma) \longrightarrow \mathrm{Ty}_{\mathbf{C}}(\Gamma')$ given by the pullback operation of $\mathbf{C}$. The axioms of a contextual category ensure that this forms a presheaf $\mathrm{Ty}_{\mathbf{C}} : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$.

More generally, by a **context extension** of $\Gamma \in \mathrm{Ob}_n\,\mathbf{C}$, we mean some object $\Gamma' \in \mathrm{Ob}_{n+m}\,\mathbf{C}$ with $\mathrm{ft}^m\,\Gamma' = \Gamma$. Again, we will write such an extension (considered as an object of $\mathbf{C}$) as $\Gamma.\Delta$, with a canonical projection $p_\Delta : \Gamma.\Delta \longrightarrow \Gamma$ obtained by composing the projections $\Gamma.\Delta \longrightarrow \mathrm{ft}(\Gamma.\Delta) \longrightarrow \ldots \longrightarrow \mathrm{ft}^m(\Gamma.\Delta) = \Gamma$. Similarly, given $f : \Gamma' \longrightarrow \Gamma$ and a context extension Δ of Γ, by iterating the pullback of types we obtain a pullback context extension $f^*\Delta$ over Γ', with $f.\Delta : \Gamma'.f^*\Delta \longrightarrow \Gamma.\Delta$.

Given $\Gamma \in \mathbf{C}$ and $A \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$, we write $\mathrm{Tm}_{\mathbf{C},\Gamma}(A)$ for the set of sections $s : \Gamma \longrightarrow \Gamma.A$ of the projection p_A. When no confusion is possible, we will omit the subscripts, writing $\mathrm{Ty}(\Gamma)$ and $\mathrm{Tm}(A)$ respectively.

Definition 2.3. Contextual categories can be equipped with additional operations corresponding to the various type-constructors of Martin-Löf Type Theory. For the present paper we consider just the structure corresponding to:

- identity types (denoted Id);
- unit types (1) and dependent sum types (Σ);
- dependent function types, with functional extensionality rules (together, Π_{ext}).

For the definitions of these structures, see [KL12, App. B].

For each choice of constructors, contextual categories with such structure are again models of an essentially algebraic theory (extending the e.a.t. of contextual categories), so have a natural notion of morphism: contextual functors preserving the extra structure.

We write $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma}$ for the category of contextual categories equipped with Id-, 1-, and Σ-types; and $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$ the category of contextual categories with all these plus extensional dependent function types. When a statement, construction, or proof can be read in parallel for each of these categories, we will refer to them as $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$. There is moreover (again, by their description as e.a.t.'s) a free–forgetful adjunction

$$\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \mathrm{CxlCat}_{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$$

Remark 2.4. One often considers other logical structure besides Id, 1, Σ, and Π_{ext}. Some of the results of this paper extend directly to such further structure; others do not. In the absence of a good general framework for such structure, however, we restrict ourselves for the present paper to the case of $\mathsf{Id}, 1, \Sigma(, \Pi_{\mathsf{ext}})$, except for a few definitions and constructions that only assume Id-types.

Definition 2.5. Following Garner [Gar09, Prop. 3.3.1], we note that Id-types on a contextual category allow the construction of more general **identity contexts**. Specifically, given $\Gamma \in \mathrm{Ob}_n\, \mathbf{C}$ and a context extension $\Gamma.\Delta \in \mathrm{Ob}_{n+m}\, \mathbf{C}$, there is a further context extension $\Gamma.\Delta.p^*_\Delta\Delta.\mathsf{Id}_\Delta \in \mathrm{Ob}_{n+3m}\, \mathbf{C}$, along with a reflexivity map and elimination operation generalizing those of the identity type $\Gamma.A.p^*_A A.\mathsf{Id}_A$ of a single type over Γ.

Definition 2.6. Given $f, g : \Gamma \longrightarrow \Delta$ in a contextual category $\mathbf{C}$, a **homotopy** H from f to g (denoted $H : f \sim g$) is a factorization of $(f, g) : \Gamma \longrightarrow \Delta \times \Delta = \Delta.p^*_\Delta\Delta$ through the identity context $\Delta.p^*_\Delta\Delta.\mathsf{Id}_\Delta \xrightarrow{p_{\mathsf{Id}_\Delta}} \Delta.p^*_\Delta\Delta$.

There are various established definitions of equivalence in contextual categories, all essentially equivalent [Uni13, Ch. 4]; we choose the following:

Definition 2.7. Let $\mathbf{C}$ be a contextual category with identity types.

- A **structured equivalence** $w : \Gamma \simeq \Delta$ consists of a map $f : \Gamma \longrightarrow \Delta$, together with maps $g_1, g_2 : \Delta \longrightarrow \Gamma$ and homotopies $\eta : f g_1 \sim 1_\Delta$ and $\varepsilon : g_2 f \sim 1_\Gamma$.
- An **equivalence** $\Gamma \xrightarrow{\sim} \Delta$ in $\mathbf{C}$ is a map $f : \Gamma \longrightarrow \Delta$ for which there exist some $g_1, g_2, \eta, \varepsilon$ making it a structured equivalence.

Definition 2.8. Given a contextual category $\mathbf{C}$ and an object $\Gamma \in \mathbf{C}$, the **fibrant slice** contextual category $\mathbf{C}/\!/\Gamma$ is given by:

- objects in $\mathrm{Ob}_m\, \mathbf{C}/\!/\Gamma$ are context extensions $\Gamma.\Delta \in \mathrm{Ob}_{n+m}\, \mathbf{C}$;
- $(\mathbf{C}/\!/\Gamma)(\Gamma.\Delta,\ \Gamma.\Delta') := (\mathbf{C}/\Gamma)(\Gamma.\Delta,\ \Gamma.\Delta')$;
- the remaining structure is inherited from $\mathbf{C}$.

If $\mathbf{C}$ carries identity types (resp. 1, Σ, Π_{ext}), then so does $\mathbf{C}/\!/\Gamma$.

This satisfies the familiar categorical property that a slice of a slice is again a slice, in that $(\mathbf{C}/\!/\Gamma)/\!/\Gamma.\Delta \cong \mathbf{C}/\!/(\Gamma.\Delta)$.

Moreover, any contextual functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ and object $\Gamma \in \mathbf{C}$ induce an evident contextual functor $F/\!/\Gamma : \mathbf{C}/\!/\Gamma \longrightarrow \mathbf{D}/\!/F\Gamma$; and this preserves any logical structure that F does.

Fibration categories

Fibration categories and variations thereof (like Shulman's **type-theoretic fibration categories** [Shu15b, Def. 2.1] and Joyal's **tribes** [Joy14]) have proven useful when studying homotopy-theoretic aspects of type theory. We begin by recalling some of the basic definitions and constructions.

Fibration categories were introduced by Brown [Bro73] as **categories of fibrant objects**. We slightly strengthen them, following other recent authors (e.g. [Szu14, Def. 1.1]).

Definition 2.9. A **fibration category** consists of a category $\mathcal{C}$ together with two wide subcategories (subcategories containing all objects): $\mathcal{F}$ of **fibrations** and $\mathcal{W}$ of **weak equivalences** such that:

(1) weak equivalences satisfy the **2-out-of-6** property; that is, given a composable triple of morphisms:
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} Z$$
if hg and gf are weak equivalences, then so are f, g, h, and hgf.
(2) all isomorphisms are **acyclic fibrations** (i.e., are both fibrations and weak equivalences).
(3) pullbacks along fibrations exist; fibrations and acyclic fibrations are stable under pullback.
(4) $\mathcal{C}$ has a terminal object 1; the canonical map $X \longrightarrow 1$ is a fibration for any object $X \in \mathcal{C}$ (that is, all objects are **fibrant**).
(5) every map can be factored as a weak equivalence followed by a fibration.

Given a fibration category $\mathcal{C}$, its **homotopy category** $\operatorname{Ho}\mathcal{C}$ is the result of formally inverting the weak equivalences. It can be described more explicitly using the notion of weak right homotopy.

Definition 2.10 ([Bro73, §2])**.** A **path object** for $X \in \mathcal{C}$ is any factorization $X \xrightarrow{\sim} PX \twoheadrightarrow X \times X$ of the diagonal map as a weak equivalence followed by a fibration.

Maps $f, g : X \longrightarrow Y$ are **weakly right homotopic**, $f \sim g$, if for some trivial fibration $t : X' \longrightarrow X$ the maps $ft, gt : X' \longrightarrow Y$ factor jointly through some path object $PY \twoheadrightarrow Y \times Y$.

$$\begin{array}{ccc} X' & \xrightarrow{h} & PY \\ {\scriptstyle t}\downarrow & & \downarrow \\ X & \xrightarrow{\langle f,g\rangle} & Y \times Y. \end{array}$$

Say f, g are **(strictly) right homotopic**, $f \sim_r g$, if one can take $X' = X$, $t = \mathrm{id}_X$.[1]

In general fibration categories, the weak notion is more important:

Theorem 2.11 ([Bro73, Thm. 1])**.** *For any fibration category $\mathcal{C}$, the homotopy category* $\operatorname{Ho}\mathcal{C}$ *may be taken as the category with the same objects as $\mathcal{C}$, and with* $\operatorname{Hom}_{\operatorname{Ho}\mathcal{C}}(X,Y) = \operatorname{Hom}_{\mathcal{C}}(X,Y)/\sim$.

Definition 2.12.

- A functor between fibration categories is **exact** if it preserves fibrations, acyclic fibrations, pullbacks along fibrations, and a terminal object.
- An exact functor is a **weak equivalence** of fibration categories if it induces an equivalence of homotopy categories.

As mentioned above, the framework of fibration categories can be used to study homotopy-theoretic aspects of type theory. Let $\mathbf{C}$ be a contextual category with Id-types. Define classes $\mathcal{W}$, $\mathcal{F}$ of maps in $\mathbf{C}$ by:

[1] In some recent literature, e.g. [Szu14], *right homotopic* is used for the weak notion; we distinguish that explicitly to avoid clashing with more established usage.

- $\mathcal{W}$ is precisely the equivalences of Definition 2.7;
- $\mathcal{F}$ consists of maps isomorphic to some composite of canonical projections.

Theorem 2.13 ([AKL15, Thm. 3.2.5], [Kap15, Ex. 2.6.(3)])**.**

(1) For any contextual category $\mathbf{C}$ with Id, 1, and Σ, these classes $\mathcal{F}$ and $\mathcal{W}$ make $\mathbf{C}$ a fibration category.

(2) This forms the object part of a faithful functor $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma} \longrightarrow \mathrm{FibCat}$.

The fibration categories arising from contextual categories are particularly nice in that all of their objects are also cofibrant; that is, every acyclic fibration admits a section [AKL15, Lem. 3.2.14]. This justifies the following description of their homotopy categories:

Lemma 2.14. *Let $\mathbf{C} \in \mathrm{CxlCat}_{\mathsf{Id},1,\Sigma}$. Then the homotopy category of $\mathbf{C}$ (regarded as a fibration category) can be described as follows:*

- *objects of $\mathrm{Ho}\,\mathbf{C}$ are the objects of $\mathbf{C}$;*
- *morphisms $\Gamma \longrightarrow \Delta$ in $\mathrm{Ho}\,\mathbf{C}$ are homotopy classes of maps in $\mathbf{C}$, in the sense of Definition 2.6.*

Proof. As shown in [AKL15, Thm. 3.2.5], the path-objects in a contextual category $\mathbf{C}$ are given exactly by the identity contexts. Homotopy in the sense of Definition 2.6 is therefore exactly right homotopy in the sense of Definition 2.10. Furthermore, since every object is cofibrant, this coincides with weak right homotopy. □

Finally, we note an indispensible (and easily overlooked) lemma: the property of being an equivalence does not depend on where one views a map.

Lemma 2.15. *Let $\mathbf{C}$ be a contextual category with identity types. A map $f : \Gamma.\Delta \longrightarrow \Gamma.\Delta'$ over Γ is an equivalence in $\mathbf{C}//\Gamma$ if and only if it is an equivalence in $\mathbf{C}$.*

Proof. More generally, let $f : Y \longrightarrow Y'$ be a map of fibrations over a base X, in any fibration category $\mathcal{C}$, with Y and Y' cofibrant. Then f is a homotopy equivalence in $\mathcal{C}$ if and only if it is one in $\mathcal{C}/X$. This follows by an argument originally due to Dold; it is given for model categories in [KP97, Thm. 6.3], but adapts directly to the present setting. □

3. Equivalences, fibrations, and cofibrations of contextual categories

In this section, we will define the three classes of maps: weak equivalences, cofibrations, and fibrations required for the left semi-model structure, as well as state the internal language conjectures.

We begin by introducing two notions of equivalence between contextual categories: type-theoretic and homotopy-theoreric, and proving that they are equivalent (Proposition 3.3). We then review the basic facts about known connections between type theory and higher category theory, and state the internal language conjectures (3.7). In the remainder of the section, we introduce notions of (trivial) fibrations and cofibrations between contextual categories, proving some their properties.

Logical and homotopy-theoretic equivalences

Definition 3.1. A map $F : \mathbf{C} \longrightarrow \mathbf{D}$ of contextual categories with Id-types is a **(type-theoretic) equivalence** if it satisfies

1. **weak type lifting**: for any $\Gamma \in \mathbf{C}$ and $A \in \mathrm{Ty}(F\Gamma)$, there exists $\bar{A} \in \mathrm{Ty}(\Gamma)$ together with an equivalence $F\bar{A} \xrightarrow{\sim} A$ over $F\Gamma$; and
2. **weak term lifting**: for any $\Gamma \in \mathbf{C}$, $A \in \mathrm{Ty}(\Gamma)$, and $a \in \mathrm{Tm}(FA)$, there exists $\bar{a} \in \mathrm{Tm}(A)$ together with an element of the identity type $e \in \mathrm{Tm}(\mathsf{Id}_{FA}(F\bar{a}, a))$.

Write $\mathcal{W}$ for the class of type-theoretic equivalences in $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathrm{ext}})}$.

From a logical perspective, this is a sort of conservativity between theories: compare e.g. the condition TY-CONS of [Hof95, §3.2.3].

Both lifting properties can in fact be strengthened:

Lemma 3.2. *Every type-theoretic equivalence $F : \mathbf{C} \xrightarrow{\sim} \mathbf{D}$ additionally satisfies*

(1) ***weak context lifting****: for any context $\Gamma \in \mathbf{C}$ and context extension $F\Gamma.\Delta$, there exists a context extension $\Gamma.\bar{\Delta}$ together with an equivalence $F(\Gamma.\bar{\Delta}) \xrightarrow{\sim} F\Gamma.\Delta$ over $F\Gamma$; and*

(2) ***weak section lifting****: for any context extension $\Gamma.\Delta \in \mathbf{C}$ and section $s : F\Gamma \longrightarrow F\Gamma.F\Delta$ of the generalized projection $p_{F\Delta}$, there exists a section $\bar{s} : \Gamma \longrightarrow \Gamma.\Delta$ of p_Δ, together with a homotopy $e : F\bar{s} \sim s$ over Γ.*

Proof. Induction on the length of the context. □

One can also use the associated fibration category (Theorem 2.13) to define equivalences in a more homotopy-theoretic way. Specifically, call a map $F : \mathbf{C} \longrightarrow \mathbf{D}$ a **homotopy-theoretic equivalence** if the induced functor $\mathrm{Ho}\, F : \mathrm{Ho}\, \mathbf{C} \longrightarrow \mathrm{Ho}\, \mathbf{D}$ is an equivalence of categories. It turns out these two definitions coincide:

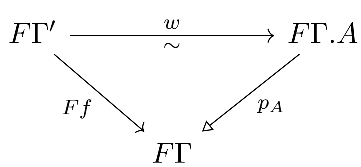
Proposition 3.3. *A contextual functor is a type-theoretic equivalence if and only if it is a homotopy-theoretic equivalence.*

Proof of Proposition 3.3. We rely on Lemma 2.14 throughout.

First, assume $F : \mathbf{C} \longrightarrow \mathbf{D}$ is a type-theoretic equivalence. Then $\mathrm{Ho}\, F$ is:

- full, by weak section lifting (Lemma 3.2(2)): a map $f : F\Gamma \longrightarrow F\Delta$ can be viewed as a section of the context extension $F\Gamma.p_\Gamma^* F\Delta \longrightarrow F\Gamma$ and as such can be lifted (up to homotopy) to a map $\bar{f} : \Gamma \longrightarrow \Gamma.\Delta$;
- faithful, by weak section lifting applied to the identity contexts;
- essentially surjective, by weak context lifting (Lemma 3.2(1)).

Conversely, assume that $\mathrm{Ho}\, F$ is an equivalence of categories. For weak type lifting, suppose $A \in \mathrm{Ty}(F\Gamma)$. Since $\mathrm{Ho}\, F$ is essentially surjective, one can find $\Gamma' \in \mathbf{C}$ and $w : F\Gamma' \xrightarrow{\sim} F\Gamma.A$. Moreover, since $\mathrm{Ho}\, F$ is full, there is some $f : \Gamma' \longrightarrow \Gamma$ such that the triangle



commutes up to homotopy. Since p_A is a fibration, we can replace w with some homotopic w' making the triangle commute on the nose. Factoring f as an equivalence u followed by a fibration $p_\Delta : \Gamma.\Delta \longrightarrow \Gamma$, and taking the iterated $\boldsymbol{\Sigma}$-type of Δ, we obtain $\bar{A} \in \mathrm{Ty}(\Gamma)$ and an equivalence $w' \cdot Fu^{-1} : F\bar{A} \xrightarrow{\sim} A$ over $F\Gamma$, as required.

Lastly, we give weak term lifting; this is a little more involved. We start by showing a "crude section lifting" property: for any $\Gamma.\Delta \in \mathbf{C}$ and section $a : F\Gamma \longrightarrow F(\Gamma.\Delta)$ of $p_{F\Delta}$, there is some section $\widehat{a} : \Gamma \longrightarrow \Gamma.\Delta$ of p_Δ, together with a homotopy $h : F\widehat{a} \sim a$ (but not yet necessarily over Γ, as required in weak term/section lifting) .

Given such $\Gamma.\Delta$ and a, by fullness of $\mathrm{Ho}\, F$ there is some map $a' : \Gamma \longrightarrow \Gamma.\Delta$ with $Fa' \sim a$. Now $F(p_\Delta a') \sim p_{F\Delta} a = \mathrm{id}_{F\Gamma}$, so by faithfulness of $\mathrm{Ho}\, F$, $p_\Delta a' \sim \mathrm{id}_\Gamma$. So since p_Γ is a fibration, we can replace a' by some section $\widehat{a}$ of p_Δ, with $\widehat{a} \sim a'$ and hence $F\widehat{a} \sim a$ as required.

Now, we can strengthen this to full term-lifting. Given Γ, A, a, take by crude section-lifting some section $\widehat{a} : \Gamma \longrightarrow \Gamma.A$ and homotopy $h : Fa' \sim a : F\Gamma \longrightarrow F\Gamma.FA$. We can split h up into $h_0 = p_{F\Gamma} h : \mathrm{id}_\Gamma \sim \mathrm{id}_\Gamma$ and $h_1 \in \mathrm{Tm}_\Gamma(\mathsf{Id}_{FA}((h_0)_! Fa', a))$, where $(h_0)_!$ denotes transport (as used in the definition of identity contexts). In type-theoretic notation,

- $x{:}F\Gamma \vdash h_0(x) : \mathsf{Id}_{F\Gamma}(x, x)$,
- $x{:}F\Gamma \vdash h_1(x) : \mathsf{Id}_{FA}(h_0(x)_! Fa'(x), a(x))$

We now apply crude section lifting to $h_0 : F\Gamma \longrightarrow F\Gamma.\mathsf{Id}_{F\Gamma}(\mathrm{id}_{F\Gamma}, \mathrm{id}_{F\Gamma})$. Type-theoretically, the result is a term $x{:}\Gamma \vdash \widehat{h_0}(x) : \mathsf{Id}_\Gamma(x, x)$, together with a homotopy $\alpha : F\widehat{h_0} \sim h_0$, which once again we split up into two parts:

- $x{:}F\Gamma \vdash \alpha_0(x) : \mathsf{Id}_{F\Gamma}(x,x)$,
- $x{:}F\Gamma \vdash \alpha_1(x) : \mathsf{Id}_{F\Gamma}(\alpha_0(x)_! F\widehat{h_0}(x), h_0(x))$.

By the J-structure (Id-elimination), one can produce a term $x, y{:}F\Gamma, u{:}\mathsf{Id}_{F\Gamma}(x,y) \vdash \theta(x,y,u) : \mathsf{Id}_{\mathsf{Id}(y,y)}(u_! F\widehat{h_0}(x), F\widehat{h_0}(y))$. [2] So, in particular, we get a term $x{:}F\Gamma \vdash \beta(x) : \mathsf{Id}_{F\Gamma}(F\widehat{h_0}(x), h_0(x))$, by composing $\theta(x,x,\alpha_0(x))^{-1}$ with $\alpha_1(x)$.

Now define the corrected lifting of a as $x{:}\Gamma \vdash \bar{a}(x) := \widehat{h_0}(x)_! \widehat{a}(x) : A$. The desired equality $x{:}F\Gamma \vdash \mathsf{Id}_{FA}(F\bar{a}(x), a(x))$ is then a composite:

$$F(\bar{a}(x)) = F(\widehat{h_0}(x))_! F(\widehat{a}(x)) \overset{\beta(x)}{=}_{\mathsf{Id}} h_0(x)_! F(\widehat{a}(x)) \overset{h_1(x)}{=}_{\mathsf{Id}} a(x). \qquad \square$$

This justifies dropping the distinction, and simply calling such functors *equivalences*. It also immediately gives:

Corollary 3.4. *$\mathcal{W}$ satisfies 2-out-of-6 and is closed under retracts.*

Proof. Equivalences of categories are closed under 2-out-of-6 and retracts; thus so are equivalences of contextual categories, as their inverse image under Ho by Proposition 3.3. □

Closing, we note another useful property:

Lemma 3.5. *If a contextual functor $F : \mathbf{C} \longrightarrow \mathbf{D}$ is an equivalence, then so is the induced functor on slices $F/\!/\Gamma : \mathbf{C}/\!/\Gamma \longrightarrow \mathbf{D}/\!/F\Gamma$.*

Proof. Straightforward, with the use of Lemma 2.15. □

Conjectures on internal languages

In this section, we will provide precise statements of the conjectures establishing dependent type theories as internal languages of (sufficiently structured) higher categories. Thus its goal is to put the results of the remainder of the paper in a broader context.

By a **category with weak equivalences**, we mean a pair $(\mathcal{C}, \mathcal{W})$, where $\mathcal{C}$ is a category and $\mathcal{W}$ a wide subcategory of $\mathcal{C}$ (whose maps we call weak equivalences). A functor F between categories with weak equivalences $(\mathcal{C}, \mathcal{W})$, $(\mathcal{C}', \mathcal{W}')$ is **homotopical** if it preserves weak equivalences. Write weCat for the category of categories with weak equivalences and homotopical functors.

Every fibration category $(\mathcal{C}, \mathcal{F}, \mathcal{W})$ has an obvious underlying category with weak equivalences $(\mathcal{C}, \mathcal{W})$, and by Ken Brown's Lemma [Hov99, Lem. 1.1.12], every exact functor is homotopical. It follows by Theorem 2.13 that every contextual category has an underlying category with weak equivalences, and this construction forms a functor $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma} \longrightarrow \mathrm{weCat}$.

The category weCat can itself be regarded as a category with weak equivalences, where the weak equivalences are Dwyer–Kan equivalences (DK-equivalences) [BK12].

Let Cat_∞ denote the full subcategory of the category sSet of simplicial sets, whose objects are quasicategories [Joy08, Def. 1.5]. We will consider Cat_∞ as a category with weak equivalences in which the weak equivalences are categorical equivalences [Joy08, Def. 1.20].

The categories with weak equivalences weCat and Cat_∞ are DK-equivalent and we will write Ho_∞ for an equivalence $\mathrm{weCat} \longrightarrow \mathrm{Cat}_\infty$. While this functor may be implemented in many ways (see [Bar16, §1.6] for several possibilities), they are all equivalent, by [Toë05, Thm. 6.3]. For concreteness, we take Ho_∞ to be the composite of the hammock localization followed by the right derived functor of the homotopy coherent nerve.

We will write $\mathrm{Cl}_\infty^{\mathsf{Id},1,\Sigma}$ for the composite functor

$$\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma} \longrightarrow \mathrm{weCat} \xrightarrow{\mathrm{Ho}_\infty} \mathrm{Cat}_\infty$$

and $\mathrm{Cl}^{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$ for its composite with the forgetful functor $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}} \longrightarrow \mathrm{CxlCat}_{\mathsf{Id},1,\Sigma}$.

For working with these functors in practice, one can exploit Szumiło's construction of the **quasicategory of frames** $\mathrm{N_f}\mathcal{C}$ in a fibration category $\mathcal{C}$ [Szu14, §3.1]. For any fibration

[2] In traditional homotopy-theoretic terms: $F\widehat{h_0}$ is a global section of the free loop space of $F\Gamma$, so must land in the *center* of $\pi_0(F\Gamma)$.

category, there is an equivalence $\mathrm{N_f}\mathcal{C} \simeq \mathrm{Ho}_\infty\,\mathcal{C}$ [KS16, Cor. 4.15]; so for fibration categories, $\mathrm{N_f}$ gives a much more explicit and workable description of the quasicategory Ho_∞ than is provided by the more general constructions.

Theorem 3.6.

(1) The functor $\mathrm{Cl}_\infty^{\mathsf{Id},1,\Sigma}$ *takes values in the category* Lex_∞ *of quasicategories with finite limits and finite limit preserving functors. Moreover, it takes equivalences of contextual categories to categorical equivalences of quasicategories.*

(2) The functor $\mathrm{Cl}_\infty^{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$ *takes values in the category* LCCC_∞ *of locally cartesian closed quasicategories and locally cartesian closed functors. As before, it takes equivalences of contextual categories to categorical equivalences of quasicategories.*

Proof. The first part of (1) is noted in [Kap15, p. 10]. For the second part, observe that the equivalences of contextual categories are exactly the weak equivalences of their underlying fibration categories, which are in turn preserved by $\mathrm{N_f}$ [Szu14, Thm. 3.3].

Similarly, the first part of (2) is exactly the statement of [Kap15, Thm. 5.8], whereas the second part follows immediately by the same reasoning as above. □

In light of the above theorem, one can formulate the following conjecture, an ∞-categorical analogue of the results of Clairambault and Dybjer [CD11], establishing intensional type theory as an internal language for suitable ∞-categories.

Conjecture 3.7. *The functors* $\mathrm{Cl}_\infty^{\mathsf{Id},1,\Sigma}$ *and* $\mathrm{Cl}_\infty^{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$ *are DK-equivalences of categories with weak equivalences.*

Ultimately, one would like to extend the above correspondences to include univalent type theories on one side and elementary ∞-toposes on the other; however, neither of these notions is yet defined. On the type-theoretic side, it is not currently clear which rules to choose, of the many proposed for univalent universes and higher inductive types. On the higher-categorical side, a precise definition of an elementary ∞-topos remains to be formulated. Lurie [Lur09] provides a detailed study of *Grothendieck* ∞-toposes, but does not pursue the idea of their elementary counterparts [Lur09, 6.1.3.11].

Once these notions have been formulated, one hopes that the functor $\mathrm{Cl}_\infty^{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$ can be promoted to a functor $\mathrm{Cl}_\infty^{\mathsf{HoTT}} : \mathrm{CxlCat}_{\mathsf{HoTT}} \longrightarrow \mathrm{ElTop}_\infty$ (where we write $\mathrm{CxlCat}_{\mathsf{HoTT}}$ for the category of contextual categories admitting rules for univalent type theories, and ElTop_∞ for the category of elementary ∞-toposes). Eventually, one hopes to obtain the following diagram, with the horizontal arrows DK-equivalences:

$$
\begin{array}{ccc}
\mathrm{CxlCat}_{\mathsf{HoTT}} & \overset{\mathrm{Cl}_\infty^{\mathsf{HoTT}}}{\underset{(\sim)}{\dashrightarrow}} & \mathrm{ElTop}_\infty \\
\downarrow & & \downarrow \\
\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}} & \overset{\mathrm{Cl}_\infty^{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}}{\underset{(\sim)}{\longrightarrow}} & \mathrm{LCCC}_\infty \\
\downarrow & & \downarrow \\
\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma} & \overset{\mathrm{Cl}_\infty^{\mathsf{Id},1,\Sigma}}{\underset{(\sim)}{\longrightarrow}} & \mathrm{Lex}_\infty
\end{array}
$$

Remark 3.8. The categories of the right hand column may be seen as the $(\infty,1)$-cores of larger $(\infty,2)$-categories. One may wonder if the maps Cl_∞ are in fact $(\infty,2)$-equivalences, for some yet-to-be-defined $(\infty,2)$-category structures on $\mathrm{CxlCat}_{(\ldots)}$.

By analogy with 1-categorical settings, we hope that this should be the case for $\mathrm{Cl}_\infty^{\mathsf{Id},1,\Sigma}$, but do not expect it for $\mathrm{Cl}_\infty^{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$ or $\mathrm{Cl}_\infty^{\mathsf{HoTT}}$. Indeed, we do not expect the full $(\infty,2)$-category LCCC_∞ to be as well-behaved at all as its $(\infty,1)$-core, essentially due to the non-covariance of exponentials.

This phenomenon appears most simply in, for example, the fact if $\mathbf{C}[A]$ is the free cartesian closed category on an object, and $F, G : \mathbf{C}[A] \longrightarrow \mathbf{D}$ are cartesian functors (determined by the objects $FA, GA \in \mathbf{D}$), then natural *isomorphisms* $\alpha : F \cong G$ are determined uniquely by isomorphisms $\alpha_A : FA \cong GA$, but no such nice property holds for general natural transformations $F \longrightarrow G$.

Fibrations and cofibrations

We define in this section two cofibrantly generated weak factorization systems, $(\mathcal{C}, \mathcal{TF})$ and $(\mathcal{A}, \mathcal{F})$ on $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$. To do so, we first set up generating sets of left maps, whose domains and codomains are presented in Definition 3.9, as freely generated objects in $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma}$. (The existence of freely generated objects follows from the presentation of $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$ as models of an essentially algebraic theory.)

Definition 3.9. Define the following freely generated objects in $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma}$:

- $\langle\!\langle\, \Gamma_n \,\rangle\!\rangle$ is freely generated by a context of length n.
- $\langle\!\langle\, \Gamma_n \vdash A \,\rangle\!\rangle$ is freely generated by a context Γ of length n, and a type A over this context. (Of course $\langle\!\langle\, \Gamma_n \vdash A \,\rangle\!\rangle \cong \langle\!\langle\, \Gamma_{n+1} \,\rangle\!\rangle$, but we distinguish them notationally for readability.)
- $\langle\!\langle\, \Gamma_n \vdash a : A \,\rangle\!\rangle$ is freely generated by Γ, A as in $\langle\!\langle\, \Gamma_n \vdash A \,\rangle\!\rangle$, and a section of p_A.
- $\langle\!\langle\, \Gamma_n \vdash A \simeq A' \,\rangle\!\rangle$ is freely generated by a context Γ of length n, types A, A' over Γ, and maps $f, g_l, g_r, \alpha_l, \alpha_r$ constituting an equivalence from A to A' over Γ.
- $\langle\!\langle\, \Gamma_n \vdash e : \mathsf{Id}_A(a, a') \,\rangle\!\rangle$ is freely generated by Γ, A as in $\langle\!\langle\, \Gamma_n \vdash A \,\rangle\!\rangle$, and a section of the composite projection map $\Gamma.A.A.\mathsf{Id}_A \longrightarrow \Gamma$ (giving all three: a, a', and e).

Applying the left adjoint functor $F : \mathrm{CxlCat}_{\mathsf{Id},1,\Sigma} \longrightarrow \mathrm{CxlCat}_{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$ gives similarly freely generated objects in $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$. When necessary for disambiguation, we may distinguish these different incarnations as e.g. $\langle\!\langle\, \Gamma_n \,\rangle\!\rangle_{\mathsf{Id},1,\Sigma}$ vs. $\langle\!\langle\, \Gamma_n \,\rangle\!\rangle_{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$; but when it is clear which category we are working in, or when statements apply to both of them, we write just $\langle\!\langle\, \Gamma_n \,\rangle\!\rangle$, and so on.

Definition 3.10. Take I and J to be the following sets of maps in $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$:

- I consists of the evident inclusions $\langle\!\langle\, \Gamma_n \,\rangle\!\rangle \longrightarrow \langle\!\langle\, \Gamma_n \vdash A \,\rangle\!\rangle$ and $\langle\!\langle\, \Gamma_n \vdash A \,\rangle\!\rangle \longrightarrow \langle\!\langle\, \Gamma_n \vdash a : A \,\rangle\!\rangle$, for all $n \in \mathbb{N}$;
- J consists of the evident inclusions $\langle\!\langle\, \Gamma_n \vdash A \,\rangle\!\rangle \longrightarrow \langle\!\langle\, \Gamma_n \vdash A \simeq A' \,\rangle\!\rangle$ and $\langle\!\langle\, \Gamma_n \vdash a : A \,\rangle\!\rangle \longrightarrow \langle\!\langle\, \Gamma_n \vdash e : \mathsf{Id}_A(a, a') \,\rangle\!\rangle$, for all $n \in \mathbb{N}$.

Definition 3.11. In each of $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma}$ and $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$, we define the classes of maps $\mathcal{TF} := I^{\pitchfork}$, $\mathcal{C} := {}^{\pitchfork}\mathcal{TF}$, $\mathcal{F} := J^{\pitchfork}$, and $\mathcal{A} := {}^{\pitchfork}\mathcal{F}$. Call maps in these classes **trivial fibrations**, **cofibrations**, **fibrations**, and **anodyne maps**.

Unwinding the universal properties of the maps in I, we see that a map $F : \mathbf{C} \longrightarrow \mathbf{D}$ is a trivial fibration just if types and terms lift along it on the nose (we will call these properties **strict type lifting** and **strict term lifting**); that is, for any $\Gamma \in \mathbf{C}$ and $A \in \mathrm{Ty}(F\Gamma)$, there is some $\bar{A} \in \mathrm{Ty}(\Gamma)$ with $F(\bar{A}) = A$, and similarly for terms. Note that since clearly strict type/term lifting implies the corresponding weak version, every trivial fibration is also a weak equivalence. These conditions are a strong form of conservativity, considered in [Lum10, Def. 4.2.5] as **contractibility**; cf. also [Hof95, Thm. 3.2.5].

Similarly, the lifting properties of a fibrations can be seen explicitly as **equivalence-lifting** and **path-lifting** respectively.

By standard results on weak factorization systems, we have:

Proposition 3.12. *$(\mathcal{C}, \mathcal{TF})$ and $(\mathcal{A}, \mathcal{F})$ are both weak factorization systems.*

The forgetful functor $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}} \longrightarrow \mathrm{CxlCat}_{\mathsf{Id},1,\Sigma}$ preserves and reflects fibrations and trivial fibrations, while its left adjoint preserves cofibrations and anodyne maps, since $I^{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$, $J^{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$ were the images of $I^{\mathsf{Id},1,\Sigma}$, $J^{\mathsf{Id},1,\Sigma}$ under the left adjoint. Note however that the forgetful functor will *not* generally preserve cofibrations or anodyne maps.

It also follows automatically that a map of contextual categories is a cofibration (resp. anodyne) just if it is a retract of a cell complex built from the basic maps in I (resp. J). We will not make formal use of this fact, but it is helpful for intuition: a typical cofibration is an I-cell complex, i.e. an extension of type theories obtained by repeatedly adjoining new types and terms (possibly infinitely many), but no new judgemental equalities. In particular, a typical cofibrant object is a type theory generated (over the constructors $\mathsf{Id}, 1, \Sigma$ or $\mathsf{Id}, 1, \Sigma, \Pi_{\mathsf{ext}}$ under consideration) just by algebraic type and term rules, with no extra definitional equalities. Similarly, a typical anodyne map is a J-cell complex, i.e. an extension built by repeatedly adjoining new terms and types along with equivalences or propositional equalities to pre-existing ones.

Proposition 3.13. *Every object in* $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$ *is fibrant.*

Proof. Immediate since the generating anodyne maps all have retractions, given by the identity equivalence and reflexivity term, respectively. □

Note that checking the equivalence-lifting criterion for a fibration directly would be rather tedious in practice, since it involves lifting *structured* equivalences. In Corollary 4.16 below, we show that this is happily unnecessary: it is enough to lift unstructured equivalences.

4. Categories with attributes

For assembling the classes of maps above into semi-model structures on $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$, our main technical workhorse will be the category $\mathbf{C}^{\mathrm{Eqv}}$ of **span-equivalences** in $\mathbf{C}$—almost a path object, but not quite—along with some related auxiliary constructions.

All these are most naturally viewed not directly as constructions on contextual categories, but as living in the slightly more general world of **categories with attributes** (CwA's).

In this section, we therefore recall and develop some background results on CwA's and their relationship with contextual categories, before tackling the span-equivalence constructions themselves in Section 5.

Categories with Attributes: background

Definition 4.1. A **category with attributes** (CwA) consists of:

(1) a category $\mathbf{C}$, with a chosen terminal object 1;
(2) a functor $\mathrm{Ty} : \mathbf{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$;
(3) an assignment to each $A \in \mathrm{Ty}(\Gamma)$, an object $\Gamma.A \in \mathbf{C}$ and a map $p_A : \Gamma.A \longrightarrow \Gamma$;
(4) for each $A \in \mathrm{Ty}(\Gamma)$ and $f : \Delta \longrightarrow \Gamma$, a map $f.A : \Delta.f^*A \longrightarrow \Gamma.A$ (called the **connecting map**) such that the following square is a pullback:

$$\begin{array}{ccc} \Delta.f^*A & \xrightarrow{f.A} & \Gamma.A \\ {\scriptstyle p_{f^*A}}\downarrow & \lrcorner & \downarrow{\scriptstyle p_A} \\ \Delta & \xrightarrow{f} & \Gamma \end{array}$$

As defined, categories with attributes are models for an evident essentially algebraic theory. A map of categories with attributes is a homomorphism of such models: explicitly, a functor $F : \mathbf{C} \longrightarrow \mathbf{C}'$ and transformation $F_{\mathrm{Ty}} : \mathrm{Ty}_{\mathbf{C}} \longrightarrow \mathrm{Ty}_{\mathbf{C}'} \cdot F$, strictly preserving all the structure (chosen terminal object, context extension, and so on).

Write CwA for the category of categories with attributes. Just as in the case of contextual categories, one may equip categories with attributes with additional structure corresponding to different type constructors. The translations of these structures from the language of contextual categories to that of categories with attributes are straightforward. We will write $\mathrm{CwA}_{\mathsf{Id},1,\Sigma}$ and $\mathrm{CwA}_{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$ for the categories of categories with attributes equipped with the corresponding extra structure, and when a statement applies to both of these cases, we will indicate it by writing $\mathrm{CwA}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$

Definition 4.2. The presheaf Ty defined in Notation 2.2 allows us to regard any contextual category as a category with attributes. This extends to an evident faithful functor $\mathrm{CxlCat} \longrightarrow \mathrm{CwA}$; and indeed exhibits CxlCat as the full subcategory consisting of CwA's equipped with a suitable grading on objects, since such a grading is unique if it exists, and is automatically preserved by any CwA map.

To go the other way, we generalize Definition 2.8:

Definition 4.3. Let $\mathbf{C}$ be a CwA, and Γ an object of $\mathbf{C}$.

A **context over** Γ is a sequence $\Delta = (A_0, \cdots, A_{n-1})$, where $A_0 \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$, …, $A_i \in \mathrm{Ty}_{\mathbf{C}}(\Gamma.A_0.\cdots.A_{i-1})$, …. Any context induces an evident **context extension** $\Gamma.\Delta := \Gamma.A_0.\cdots.A_{n-1}$, with projection map $p_\Delta : \Gamma.\Delta \longrightarrow \Gamma$.

The **fibrant slice** $\mathbf{C}/\!/\Gamma$ is the contextual category defined as follows:

(1) objects of degree n are contexts Δ of length n over Γ;
(2) $\mathbf{C}/\!/\Gamma(\Delta, \Delta') := \mathbf{C}/\Gamma(\Gamma.\Delta, \Gamma.\Delta')$, and the category structure is inherited from $\mathbf{C}/\Gamma$;
(3) reindexing and the connecting maps are inherited directly from $\mathbf{C}$.

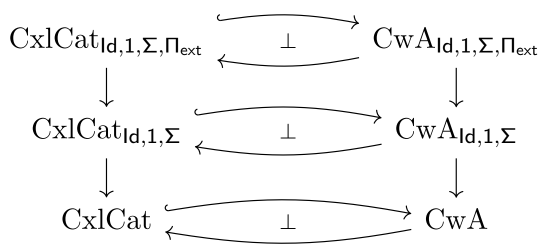
Moreover, an Id-type (resp. 1, Σ, extensional Π-type) structure on $\mathbf{C}$ induces one on $\mathbf{C}/\!/\Gamma$.

A map $f : \Gamma' \longrightarrow \Gamma$ in $\mathbf{C}$ induces a contextual functor $f^* : \mathbf{C}/\!/\Gamma \longrightarrow \mathbf{C}/\!/\Gamma'$, functorially in f, and preserving all logical structure under consideration.

Similarly, for any CwA map $F : \mathbf{C} \longrightarrow \mathbf{D}$ and object $\Gamma \in \mathbf{C}$, there is as before an induced slice functor $F/\!/\Gamma : \mathbf{C}/\!/\Gamma \longrightarrow \mathbf{D}/\!/F\Gamma$, preserving any logical structure that F does.

In particular, we call $\mathbf{C}/\!/1$ the **contextual core** of $\mathbf{C}$, and denote this by $\mathrm{core}\,\mathbf{C}$.

Proposition 4.4. *This forms the object part of a functor* $\mathrm{core} : \mathrm{CwA} \longrightarrow \mathrm{CxlCat}$*, right adjoint to the inclusion functor (so exhibiting* CxlCat *as a coreflective subcategory of* CwA*), and similarly for the categories with* $\mathsf{Id}, 1, \Sigma$ *and* $\mathsf{Id}, 1, \Sigma, \Pi_{\mathsf{ext}}$*:*



Proof. Given $F : \mathbf{C} \longrightarrow \mathbf{D}$, with $\mathbf{C}$ a contextual category and $\mathbf{D}$ an arbitrary CwA, F factors uniquely through $\mathbf{D}/\!/1 \longrightarrow \mathbf{D}$ via a contextual map $\bar{F} : \mathbf{C} \longrightarrow \mathbf{D}/\!/1$, since every $\Gamma \in \mathrm{Ob}_n\,\mathbf{C}$ is uniquely expressible in the form $1.A_0 \ldots A_{n-1}$, and hence must be sent under $\bar{F}$ to the sequence $(FA_0, \ldots, FA_n)$.

It is similarly routine to check that $\mathbf{D}/\!/1 \longrightarrow \mathbf{D}$ preserves all logical structure under consideration, and given F as above, $\bar{F}$ preserve such structure if and only if F does. $\square$

Equivalences in CwAs

In a contextual category with identity types, Definition 2.7 gives a good notion of when a map is an equivalence.

In a category with attributes, however, objects are not in general built up out of types (i.e., there may objects $\Gamma \in \mathbf{C}$ whose canonical map $\Gamma \longrightarrow 1$ cannot be written as a composite of p-maps). So we do not have identity contexts for arbitrary objects, nor hence a notion of homotopy between arbitrary maps; so we need a slightly less direct definition of equivalences.

Definition 4.5. Let $\mathbf{C}$ be a CwA with Id-types. A map $\Gamma \longrightarrow \Delta$ in $\mathbf{C}$ an **equivalence** if the induced contextual functor $f^* : \mathbf{C}/\!/\Delta \longrightarrow \mathbf{C}/\!/\Gamma$ is an equivalence of contextual categories (in the sense of Definition 3.1).

When $\mathbf{C}$ is contextual, or more generally when f lies in some fibrant slice of $\mathbf{C}$, this coincides with the established definition:

Proposition 4.6. *Let* $\mathbf{C}$ *be a CwA with* Id*-types, and* $f : \Gamma.\Delta_1 \longrightarrow \Gamma.\Delta_2$ *a map between context extensions over* $\Gamma \in \mathbf{C}$*. Then* f *is an equivalence in the sense of Definition 4.5 if and only if, considered as a map* $\Delta_1 \longrightarrow \Delta_2$ *in the contextual category* $\mathbf{C}/\!/\Gamma$*, it is an equivalence in the sense of Definition 2.7.*

Proof. ($\Rightarrow$): First, take the canonical element $\Gamma,\, y_1{:}\Delta_1 \vdash y_1 : \Delta_1$ and lift this along f^* (by Lemma 3.2) to get $\Gamma,\, y_2{:}\Delta_2 \vdash g(y_2) : \Delta_1$ with $\Gamma,\, y_1{:}\Delta_1 \vdash \eta(y_1) : \mathsf{Id}_{\Delta_1}(gf(y_1), y_1)$. This defines $g\colon \Delta_2 \longrightarrow \Delta_1$ in $\mathbf{C}/\!/\Gamma$ and shows that it is a one-sided quasi-inverse of f.

Next, consider the identity context $\Gamma,\, y_2{:}\Delta_2 \vdash \mathsf{Id}_{\Delta_2}(f(g(y_2)), y_2)$. Pulled back along f, it gives $\Gamma,\, y_1{:}\Delta_1 \vdash \mathsf{Id}_{\Delta_1}(fgf(y_1), f(y_1))$, which is inhabited by the action of f on $\eta(y_1)$. Lifting this along f^* shows that g is a quasi-inverse for f.

($\Leftarrow$): Assuming that f is equivalence, we may choose its quasi-inverse g. The (weak) lifts of types and terms are then defined by pullback along g and the required equivalences/paths are given by the homotopies $fg \sim \mathrm{id}$ and $gf \sim \mathrm{id}$. □

(Note that in the argument for ($\Rightarrow$), we did not use the weak type lifting property along f^*, only the weak term lifting; so as a scholium we see that for functors of the form $f^* : \mathbf{C}/\!/\Gamma.\Delta \longrightarrow \mathbf{C}/\!/\Gamma.\Delta'$, where f is a map over Γ, weak term lifting implies weak type lifting.)

Proposition 4.7. *Equivalences in a CwA satisfy 2-out-of-6, and are stable under retracts.*

Proof. By the same properties for equivalences of contextual categories (Corollary 3.4). □

Proposition 4.8. *Suppose Δ_1, Δ_2 are context extensions of Γ, $w : \Gamma.\Delta_1 \longrightarrow \Gamma.\Delta_2$ is a map over Γ, and $f : \Gamma' \longrightarrow \Gamma$ is any map. If w is an equivalence (in $\mathbf{C}$), then so is $f^*w : f^*\Delta_1 \longrightarrow f^*\Delta_2$.*

Proof. We can view w as a map in $\mathbf{C}/\!/\Gamma$; by Proposition 4.6, it is an equivalence there in the contextual sense. But $f^* : \mathbf{C}/\!/\Gamma \longrightarrow \mathbf{C}/\!/\Gamma'$ is a contextual functor preserving Id-types; so f^*w is an equivalence in $\mathbf{C}/\!/\Gamma'$, and hence in $\mathbf{C}$. □

Proposition 4.9. *If $f : \Gamma' \longrightarrow \Gamma$ is an equivalence, and $A \in \mathrm{Ty}(\Gamma)$, then $f.A : \Gamma'.f^*A \longrightarrow \Gamma.A$ (the pullback of f along p_A) is again an equivalence.*

Proof. $(f.A)^*$ is equal (on the nose!) to the functor $f^*/\!/p_A$, which is an equivalence by Lemma 3.5. □

This can be seen as a form of right properness for equivalences in a CwA.

Fibrations and cofibrations of CwAs

In this section, we define classes of fibrations between CwAs analogously to how they were defined for contextual categories in Section 3. As before, we start with the generating sets of left maps in $\mathrm{CwA}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$, generalizing Definition 3.9.

Definition 4.10.

- $\langle\!\langle\, \Gamma \,\rangle\!\rangle^{\mathrm{CwA}}$ is freely generated (as a CwA with Id, 1, Σ, and possibly Π_{ext}) by a single object $\Gamma \in \mathbf{C}$.
- $\langle\!\langle\, \Gamma \vdash A \,\rangle\!\rangle^{\mathrm{CwA}}$ is freely generated by $\Gamma \in \mathbf{C}$ and $A \in \mathrm{Ty}(\Gamma)$.
- $\langle\!\langle\, \Gamma \vdash a : A \,\rangle\!\rangle^{\mathrm{CwA}}$ is freely generated by Γ, A as above, and a section a of $p_A : \Gamma.A \longrightarrow \Gamma$.
- $\langle\!\langle\, \Gamma \vdash A \simeq A' \,\rangle\!\rangle^{\mathrm{CwA}}$ is freely generated by a context Γ, types $A, A' \in \mathrm{Ty}(\Gamma)$, and maps $f, g_l, g_r, \alpha_l, \alpha_r$ constituting an equivalence $\Gamma.A \simeq \Gamma.A'$ in $\mathbf{C}/\!/\Gamma$.
- $\langle\!\langle\, \Gamma \vdash e : \mathsf{Id}_A(a, a') \,\rangle\!\rangle^{\mathrm{CwA}}$ is freely generated by Γ, A as above, and a section of the iterated projection $\Gamma.A.A.\mathsf{Id}_A \longrightarrow \Gamma$.

Again, we disambiguate as e.g. $\langle\!\langle\, \Gamma \vdash A \,\rangle\!\rangle^{\mathrm{CwA}}_{\mathsf{Id},1,\Sigma}$ when necessary; but it is never necessary.

Definition 4.11. Take I and J to be the following sets of maps in $\mathrm{CwA}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$:

- I consists of the inclusions $\langle\!\langle\, \Gamma \,\rangle\!\rangle^{\mathrm{CwA}} \longrightarrow \langle\!\langle\, \Gamma \vdash A \,\rangle\!\rangle^{\mathrm{CwA}}$ and $\langle\!\langle\, \Gamma \vdash A \,\rangle\!\rangle^{\mathrm{CwA}} \longrightarrow \langle\!\langle\, \Gamma \vdash a : A \,\rangle\!\rangle^{\mathrm{CwA}}$;
- J consists of the inclusions $\langle\!\langle\, \Gamma \vdash A \,\rangle\!\rangle^{\mathrm{CwA}} \longrightarrow \langle\!\langle\, \Gamma \vdash A \simeq A' \,\rangle\!\rangle^{\mathrm{CwA}}$ and $\langle\!\langle\, \Gamma \vdash a : A \,\rangle\!\rangle^{\mathrm{CwA}} \longrightarrow \langle\!\langle\, \Gamma \vdash p : \mathsf{Id}_A(a, a') \,\rangle\!\rangle^{\mathrm{CwA}}$.

Thus the sets I and J contain just two maps each, in contrast with the generating left maps for $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$, where we required infinitely many maps due to the grading of objects.

Definition 4.12. A map $F : \mathbf{C} \longrightarrow \mathbf{D}$ in $\mathrm{CwA}_{\mathsf{Id},1,\Sigma}$ (resp. $\mathrm{CwA}_{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$) is a **local fibration** (resp. **local trivial fibration**) if it is right-orthogonal to the maps I (resp. J) of Definition 4.11.

Just as in the case of contextual categories (Definition 3.11), one may unwind this orthogonality to describe local (trivial) fibrations explicitly in terms of type/term lifting. Specifically, a map $F : \mathbf{C} \longrightarrow \mathbf{D}$ of CwA's is a local fibration exactly when

- given any $\Gamma \in \mathbf{C}$, $A \in \mathrm{Ty}_{\mathbf{C}}\Gamma$, $B \in \mathrm{Ty}_{\mathbf{D}}(F\Gamma)$, and structured equivalence $w : FA \simeq B$ over $F\Gamma$, there exists a lift $\bar{B} \in \mathrm{Ty}_{\mathbf{C}}\Gamma$ together with a structured equivalence $\bar{w} : A \simeq \bar{B}$ over Γ such that $F\bar{w} = w$;
- given any $\Gamma \in \mathbf{C}$, $A \in \mathrm{Ty}_{\mathbf{C}}\Gamma$, a section a of the projection p_A in $\mathbf{C}$, a section a' of p_{FA} in $\mathbf{D}$, and a section e of $p_{\mathsf{Id}_{FA}(Fa,a')}$, there exist lifts of a', e to $\mathbf{C}$.

and a local trivial fibration just when types and terms lift along it on the nose.

Several useful facts follow immediately from this description:

Proposition 4.13.

(1) a contextual functor is a (trivial) fibration in the sense of Definition 3.11 exactly if, viewed as a map of CwA's, it is a local (trivial) fibration;

(2) a map $F : \mathbf{C} \longrightarrow \mathbf{D}$ of CwA's is a local (trivial) fibration exactly if all its slice functors $F/\!/\Gamma : \mathbf{C}/\!/\Gamma \longrightarrow \mathbf{D}/\!/F\Gamma$ are (trivial) fibrations;

(3) the functors core $: \mathrm{CwA}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})} \longrightarrow \mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$ *send local (trivial) fibrations to (trivial) fibrations;*

(4) the inclusion functors $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})} \longrightarrow \mathrm{CwA}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$ perserve the corresponding left classes (by adjunction from the previous statement). □

Remark 4.14. Note that we use the word *local* here in the sense of a property defined slice-wise, rather than in its more common homotopy-theoretic sense of a property defined homset-wise. We avoid calling them just "(trivial) fibrations" since they do not behave the way one would expect such classes to behave, as their lifting properties are only for terms and types, not for arbitrary objects. In particular, local trivial fibrations do not satisfy "relative 2-out-of-3" among local fibrations. To see this, consider some map $F : \mathbf{C} \longrightarrow \mathbf{C}'$, and another CwA $\mathbf{D}$. Then the inclusions $\mathbf{D} \longrightarrow \mathbf{C} + \mathbf{D}$ and $\mathbf{D} \longrightarrow \mathbf{C}' + \mathbf{D}$ are local trivial fibrations, and $F + \mathrm{id}_{\mathbf{D}} : \mathbf{C} + \mathbf{D} \longrightarrow \mathbf{C}' + \mathbf{D}$ is a local fibration; but $F + \mathrm{id}_{\mathbf{D}}$ is a local trivial fibration only if F was.

As mentioned at the end of Section 3, the definition of a (local) fibration is slightly tedious to check directly, as it involves lifting structured equivalences. Happily, this can be simplified:

Lemma 4.15. *Let $F : \mathbf{C} \longrightarrow \mathbf{D}$ be a map of CwA's with Id-types, satisfying the path-lifting property for local fibrations, i.e. orthogonal to $\langle\!\langle\, \Gamma \vdash a : A \,\rangle\!\rangle^{\mathrm{CwA}} \longrightarrow \langle\!\langle\, \Gamma \vdash p : \mathsf{Id}_A(a, a') \,\rangle\!\rangle^{\mathrm{CwA}}$. Then the following are equivalent:*

(a) f satisfies the equivalence-lifting property of a local fibration, i.e. is orthogonal to $\langle\!\langle\, \Gamma \vdash A \,\rangle\!\rangle^{\mathrm{CwA}} \longrightarrow \langle\!\langle\, \Gamma \vdash A \simeq A' \,\rangle\!\rangle^{\mathrm{CwA}}$;

(b) for any $A \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$, $A' \in \mathrm{Ty}_{\mathbf{D}}(F\Gamma)$, and (unstructured) equivalence $f : F\Gamma.FA \xrightarrow{\sim} F\Gamma.A'$ over Γ, there are lifts $\bar{A}' \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$ of A' and $\bar{f} : \Gamma.A \xrightarrow{\sim} \Gamma.\bar{A}'$ of f;

(c) for any $A \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$, $A' \in \mathrm{Ty}_{\mathbf{D}}(F\Gamma)$, and (unstructured) equivalence $g : F\Gamma.A' \xrightarrow{\sim} F\Gamma.FA$ over Γ, there are lifts $\bar{A}' \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$ of A' and $\bar{g} : \Gamma.\bar{A}' \xrightarrow{\sim} \Gamma.A$ of g.

Proof. (a) ⇒ (b), (c) is immediate.

(b) ⇒ (a): given $A \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$, $A' \in \mathrm{Ty}_{\mathbf{D}}(F\Gamma)$, and a structured equivalence $(f, g_1, \eta, g_2, \varepsilon)$ from FA to A' over $F\Gamma$, we need to lift the whole structured equivalence, on the nose. By (b), f lifts to an equivalence $\bar{f} : A \longrightarrow \bar{A}'$, for which we may choose weak inverse data

$(g'_l, \eta', g'_2, \varepsilon')$. Then $(Fg'_l, F\eta', Fg'_2, F\varepsilon')$ give alternate weak inverse data for f. By essential uniqueness of such data, Fg'_l is propositionally equal to g_1, so by the path-lifting property, we can lift g_1 (and the connecting equality) on the nose, and similarly for η, g_2, ε in turn.

(c) $\Rightarrow$ (b): suppose $g : F\Gamma.A' \xrightarrow{\sim} F\Gamma.FA$ is as in (b). Choose some weak inverse $f : F\Gamma.FA \longrightarrow F\Gamma.A'$ for g over Γ. By (c), lift A', f to some $\bar{A}' \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$, $\bar{f} : \Gamma.A \xrightarrow{\sim} \Gamma.\bar{A}'$. Choose some weak inverse g' for $\bar{f}$. Now g and Fg' are both weak inverses for f, so are propositionally equal; so by the path-lifting property, we can lift g, as desired. □

Corollary 4.16. *A map $F : \mathbf{C} \longrightarrow \mathbf{D}$ of CwA's with* Id*-types is a local fibration (so, if $\mathbf{C}$, $\mathbf{D}$ are contextual, a fibration) if and only if it satisfies the path-lifting property from the definition, together with any one of the equivalent equivalence-lifting properties of Lemma 4.15.*

5. The Reedy span-equivalences construction

In this section, from a given CwA $\mathbf{C}$ with Id-types, we construct three new CwA's:

- $\mathbf{C}^{\mathrm{Eqv}}$, the **span-equivalences** in $\mathbf{C}$;
- $\mathbf{C}^{\mathrm{EqvRefl}}$, the category of **trivial auto-(span-)equivalences** in $\mathbf{C}$;
- $\mathbf{C}^{\mathrm{EqvComp}}$, the category of **homotopy-commutative triangles of (span-)equivalences**.

Each of these is constructed as the CwA of homotopical diagrams in $\mathbf{C}$ on a suitable inverse category. Recall that a **homotopical category** is a category with weak equivalence, satisfying the 2-out-of-6 property. A homotopical diagram in a CwA $\mathbf{C}$ is therefore a functor from a small homotopical category $(\mathcal{I}, \mathcal{W})$ to $\mathbf{C}$ taking $\mathcal{W}$ to the equivalences in $\mathbf{C}$ in the sense of the Definition 4.5.

The general construction of CwA's of homotopical diagrams on inverse categories, and logical structure on them, will be given in [KL16]. The types in these CwA's are analogous to **Reedy fibrations** of diagrams in a fibration category; their construction is thus in large part translating constructions of [Shu15b] from the language of fibration categories to the language of CwA's (and more generally comprehension categories).

For each of our three constructions, we therefore set up the appropriate homotopical inverse category on which to take diagrams; give an explicit description of the resulting CwA; and note a few facts about the result.

Precisely, the facts from [KL16] we require are:

Proposition 5.1 ([KL16], forthcoming)**.**

(1) *For any CwA $\mathbf{C}$ with* Id*-types, and any homotopical inverse category $(\mathcal{I}, \mathcal{W})$, there is a CwA $\mathbf{C}^{\mathcal{I}}_h$, whose objects are homotopical $\mathcal{I}$-diagrams in $\mathbf{C}$, and whose types are "homotopical Reedy $\mathcal{I}$-types" in $\mathbf{C}$.*

(The general construction of $\mathbf{C}^{\mathcal{I}}_h$ is somewhat involved to state; in the cases we use, we recall the resulting CwA explicitly.)

(2) *$\mathbf{C}^{\mathcal{I}}$ carries* Id*-types; and if $\mathbf{C}$ carries* 1*- and* Σ*-types, so does $\mathbf{C}^{\mathcal{I}}$.*

(3) *If $\mathbf{C}$ carries* Π_{ext}*-types, and additionally all maps of $\mathcal{I}$ are equivalences and there is a set of epis that generates all equivalences under 2-out-of-3, then $\mathbf{C}^{\mathcal{I}}$ carries* Π_{ext}*-types.*

(4) *A CwA map $F : \mathbf{C} \longrightarrow \mathbf{D}$ induces a CwA map $F^{\mathcal{I}} : \mathbf{C}^{\mathcal{I}} \longrightarrow \mathbf{D}^{\mathcal{I}}$, preserving whatever logical structure F preserved, and functorially in F.*

(5) *A homotopical discrete opfibration $f : \mathcal{I} \longrightarrow \mathcal{J}$ induces a map $\mathbf{C}^f : \mathbf{C}^{\mathcal{J}} \longrightarrow \mathbf{C}^{\mathcal{I}}$, preserving all logical structure, and functorially in F,*

(6) *If $f : \mathcal{I} \longrightarrow \mathcal{J}$ is moreover injective, then $\mathbf{C}^f$ is a local fibration; and if f is a simple equivalence, then $\mathbf{C}^f$ is a local trivial fibration.*

Remark 5.2. In the individual instances we consider, the proofs of the above results are all straightforward verifications, albeit rather lengthy. As such, we originally planned to give them individually in the present paper, before realizing they were sufficiently tedious that it was better to develop the construction in generality.

For the whole of this section, fix some CwA $\mathbf{C}$ with Id-types.

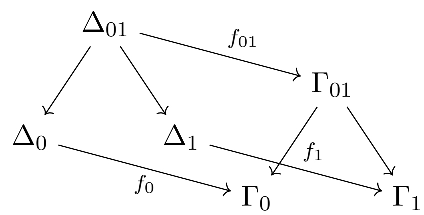
The CwA of Reedy Spans

Definition 5.3. Span is the inverse category $(0) \longleftarrow (01) \longrightarrow (1)$.

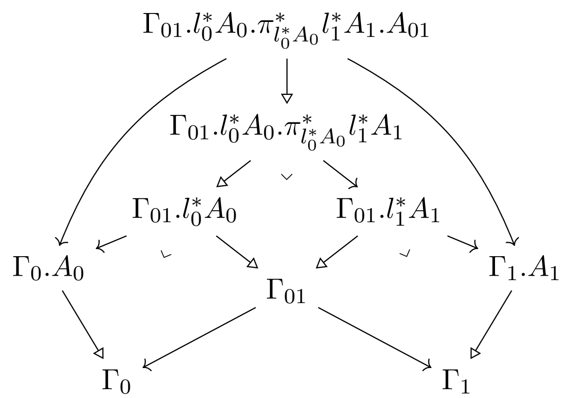
Eqv is Span considered as a homotopical category, with all maps considered as weak equivalences.

Definition 5.4. $\mathbf{C}^{\mathrm{Eqv}}$ is the CwA of homotopical diagrams on Eqv in $\mathbf{C}$. Concretely, it can be described as follows:

- objects $\vec{\Gamma}$ are "span-equivalences": spans $\Gamma_0 \xleftarrow{l_0} \Gamma_{01} \xrightarrow{l_1} \Gamma_1$ in $\mathbf{C}$, in which both l_0, l_1 are equivalences (in the sense of Definition 4.5;
- maps $\vec{f} : \vec{\Delta} \longrightarrow \vec{\Gamma}$ are natural transformations between span-equivalences



- types over an object $\vec{\Gamma} = (\Gamma_0 \xleftarrow{l_0} \Gamma_{01} \xrightarrow{l_1} \Gamma_1)$ are triples $\vec{A} = (A_0, A_1, A_{01})$, where $A_0 \in \mathrm{Ty}(\Gamma_0)$, $A_1 \in \mathrm{Ty}(\Gamma_1)$, and $A_{01} \in \mathrm{Ty}(\Gamma_{01}.l_0^* A_0.\pi^*_{l_0^* A_0} l_1^* A_1)$, with the context extension $(\Gamma_0 \xleftarrow{l_0} \Gamma_{01} \xrightarrow{l_1} \Gamma_1).(A_0, A_1, A_{01})$ and projection map as given by the following diagram:



  such that the resulting context extension is again a span-equivalence, or equivalently such that the maps $\Gamma_{01}.l_0^* A_0.\pi^*_{l_0^* A_0} l_1^* A_1.A_{01} \longrightarrow \Gamma_{01}.l_i^* A_i$ are both equivalences.
- the reindexing of a type (A_0, A_1, A_{01}) along a map (f_0, f_1, f_{01}) as in the diagram above is taken to be $(f_0^* A_0,\ f_1^* A_1,\ (f_{01}.l_0^* A_0.\pi^*_{l_0^* A_0} l_1^* A_1)^* A_{01})$, with the connecting map $(f_0, f_1, f_{01}).(A_0, A_1, A_{01})$ taken as $(f_0.A_0,\ f_1.A_1,\ (f_{01}.l_0^* A_0.\pi^*_{l_0^* A_0} l_1^* A_1).A_{01})$.

There are evident forgetful functors $P_0, P_1 : \mathbf{C}^{\mathrm{Eqv}} \longrightarrow \mathbf{C}$, taking a span to its left and right feet respectively; and since the structure on these components is defined pointwise, P_0 and P_1 are moreover maps of CwA's.

Remark 5.5. In more syntactic language, a closed type of $\mathbf{C}^{\mathrm{Span}}$ consists of three closed types in $\mathbf{C}$:

$$1 \vdash A_0 \ \mathsf{type} \qquad 1 \vdash A_1 \ \mathsf{type} \qquad x_0{:}A_0,\ x_1{:}A_1 \vdash A_{01} \ \mathsf{type}$$

More generally, a type over a context $\vec{\Gamma}$ consists of three types of the original model

$$\Gamma_0 \vdash A_0 \ \mathsf{type} \qquad \Gamma_1 \vdash A_1 \ \mathsf{type} \qquad \Gamma_{01},\ x_0{:}l_0^* A_0,\ x_1{:}l_1^* A_1 \vdash A_{01} \ \mathsf{type}$$

and the context extension is the evident span of projections

$$\Gamma_0,\ x_0{:}A_0 \ \longleftarrow\ \Gamma_{01},\ x_0{:}l_0^* A_0,\ x_1{:}l_1^* A_1,\ x_{01}{:}A_{01} \ \longrightarrow\ \Gamma_1,\ x_1{:}A_1.$$

Such a span is a **(span-)equivalence**—so lies in $\mathbf{C}^{\mathrm{Eqv}}$—exactly if it additionally satisfies the judgements that the context extensions

$$\Gamma_{01},\, x_0{:}l_0^* A_0 \vdash (x_1{:}l_1^* A_1,\, x_{01}{:}A_{01})\ \mathsf{cxt}$$
$$\Gamma_{01},\, x_1{:}l_1^* A_1 \vdash (x_0{:}l_0^* A_0,\, x_{01}{:}A_{01})\ \mathsf{cxt}$$

are both contractible (where contractibility of context extensions is defined in the evident way using their identity contexts).

Remark 5.6. A closely related model is studied by Tonelli [Ton13]. There, it is given syntactically, as the **relation** model of type theory. Precisely, Tonelli's model may be seen as the contextual core of the CwA ${\mathbf{C}_{\mathbf{T}}}^{\mathrm{Span}}$, where $\mathbf{C}_{\mathbf{T}}$ is the syntactic category of the type theory set out there.

Remark 5.7. It may seem surprising that we use the mere *property* of being an equivalence, rather than equipping the maps involved with data witnessing this.

One certainly could try building a CwA of such structured equivalences (and that would obviate the need to use spans). However, the present approach seems to simplify many proofs and constructions, since everything fits into the general framework of homotopical inverse diagrams; for instance, all logical structure is simply inherited from $\mathbf{C}^{\mathrm{Span}}$.

This approach also ensures that $\mathbf{C}^{\mathrm{Eqv}}$ depends just on the class of equivalences in $\mathbf{C}$, not on the specific choice of Id-types. This is not needed for the purposes of the present paper, but may (we expect) be useful in other applications.

Proposition 5.8.

- $\mathbf{C}^{\mathrm{Eqv}}$ *is naturally equipped with* Id*-types;*
- *if* $\mathbf{C}$ *additionally carries* Σ*- types (resp. unit types) then so does* $\mathbf{C}^{\mathrm{Eqv}}$*;*
- *if* $\mathbf{C}$ *has* Π*-types and functional extensionality, then so does* $\mathbf{C}^{\mathrm{Eqv}}$*;*
- *moreover, in all these cases, the maps* $P_i : \mathbf{C}^{\mathrm{Eqv}} \longrightarrow \mathbf{C}$ *preserve such structure.*

Proof. Id, 1, and Σ are immediate from Proposition 5.1. For Π_{ext}, we need a set of epis in Eqv generating the equivalences under 2-out-of-3; but since Eqv is posetal, all maps are epis. □

Remark 5.9. A direct construction of the structure for Proposition 5.8 consists roughly of showing that each constructor preserves equivalences of types. This is why extensionality is required for the Π-types.

Proposition 5.10. *The evident map* $\mathbf{C}^{\mathrm{Eqv}} \longrightarrow \mathbf{C} \times \mathbf{C}$ *is a local fibration of CwA's, preserving whatever logical structure is present. Similarly, the maps* $P_i : \mathbf{C}^{\mathrm{Eqv}} \longrightarrow \mathbf{C}$ *are local trivial fibrations preserving the logical structure.*

Proof. Again, an immediate application of Proposition 5.1, noting for the second part that the inclusion of either (0) or (1) into Eqv is a simple equivalence. □

Reflexivity spans

We would like to use $\mathbf{C}^{\mathrm{Eqv}}$ as some kind of path object construction. Most notions of "path object", however, include at least a "reflexivity" map $\mathbf{C} \longrightarrow P\mathbf{C}$ over the diagonal $\mathbf{C} \longrightarrow \mathbf{C} \times \mathbf{C}$; and unfortunately, $\mathbf{C}^{\mathrm{Eqv}}$ does not in general seem to admit such a map.[3] There is an evident functor on underlying categories, sending an object to the constant span on it; and this lifts suitably to a map on the presheaves of types, sending a type to its identity type span. However, this commutes only laxly with context extension, and does not commute at all with the logical structure; so it does not define a map of CwA's, let alone structured ones.

[3] We do not know of any obstruction to the existence of such a map; but it seems unlikely to us that such maps exist in general.

In lieu of a reflexivity map, therefore, we instead give reflexivity as a "weak map"; that is, a span whose left leg is a local trivial fibration:

$$\begin{array}{ccc} \mathbf{C}^{\text{EqvRefl}} & \longrightarrow & \mathbf{C}^{\text{Eqv}} \\ \downarrow & & \downarrow \\ \mathbf{C} & \xrightarrow{\Delta} & \mathbf{C} \times \mathbf{C} \end{array}$$

This suffices for the purposes of Section 6 below (and for various other applications).

Roughly speaking, a type in $\mathbf{C}^{\text{EqvRefl}}$ consists of a type A_0 of $\mathbf{C}$ equipped with an auto-(span-)equivalence $A_* \rightrightarrows A_0$ that is in some sense trivial, i.e. homotopic to the identity equivalence.

One's first thought might be to express triviality of the auto-equivalence by a reflexivity map $r : A_0 \longrightarrow A_*$ over Δ_A. However, this does not (it seems) yield a CwA; so once again, we replace this map by a weak map.

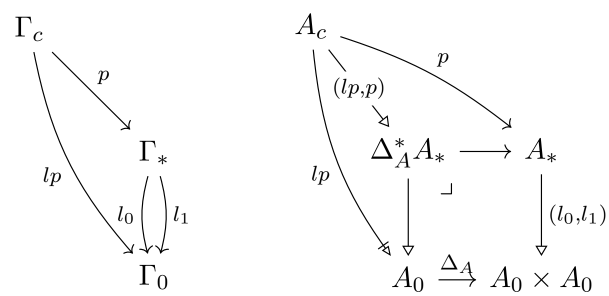
Precisely, $\mathbf{C}^{\text{EqvRefl}}$ is constructed as another CwA of homotopical inverse diagrams:

Definition 5.11. EqvRefl is the homotopical inverse category

$$C \xrightarrow{p} * \underset{l_1}{\overset{l_0}{\rightrightarrows}} 0 \qquad l_0 p = l_1 p$$

with all maps equivalences. (We write lp for the common composite $l_0 p = l_1 p$.)

Definition 5.12. $\mathbf{C}^{\text{EqvRefl}}$ is the CwA of homotopical diagrams on EqvRefl in $\mathbf{C}$. Call such diagrams **trivial auto-(span-)equivalences** in $\mathbf{C}$



General object Closed type

Remark 5.13. In traditional type-theoretic notation, suppose $\vec{\Gamma}$ is a diagram on EqvRefl:

$$\Gamma_c \xrightarrow{p} \Gamma_* \underset{l_1}{\overset{l_0}{\rightrightarrows}} \Gamma_0$$

Then a type over $\vec{\Gamma}$ in $\mathbf{C}^{\text{EqvComp}}$ consists of types

- $\Gamma_0 \vdash A_0$ type
- $\Gamma_*,\ x_0{:}l_0^* A_0,\ x_1{:}l_1^* A_0 \vdash A_*(x_0, x_1)$ type
- $\Gamma_c,\ x_0{:}(lp)^* A_0,\ x_*{:}X_*(x_0, x_0) \vdash A_c(x_0, x_*)$ type

such that the following context extensions are contractible:

- $\Gamma_*,\ x_0{:}l_0^* A_0 \vdash x_1{:}l_1^* A_0,\ x_*{:}A_*(x_0, x_1)$ cxt
- $\Gamma_c,\ x_0{:}(lp)^* A_0 \vdash x_*{:}X_*(x_0, x_0),\ x_c{:}A_c(x_0, x_*)$ cxt.

(Contractibility of these contexts corresponds to $\Gamma.A$ sending p_0 and lp to equivalences; this suffices for homotopicality since these maps generate the equivalences of EqvRefl under 2-out-of-3.)

Example 5.14. Any type $A \in \mathrm{Ty}_{\mathbf{C}}(\Gamma)$ gives rise to a type in $\mathbf{C}^{\text{EqvRefl}}$ over the constant diagram on Γ:

- $\Gamma \vdash A$ type
- $\Gamma,\ x_0, x_1{:}A \vdash \mathsf{Id}_A(x_0, x_1)$ type

- $\Gamma, x_0{:}A, x_*{:}\mathsf{Id}_A(x_0,x_0) \vdash \mathsf{Id}_{\mathsf{Id}_A(x_0,x_0)}(x_*, \mathsf{r}(x_0))$ type

Proposition 5.15.

- $\mathbf{C}^{\mathrm{EqvRefl}}$ *carries* Id*-types;*
- *if* $\mathbf{C}$ *additionally carries* Σ*- types (resp. unit types) then so does* $\mathbf{C}^{\mathrm{EqvRefl}}$*;*
- *if* $\mathbf{C}$ *has* Π*-types and functional extensionality, then so does* $\mathbf{C}^{\mathrm{Eqv}}$*;*
- *moreover, in all these cases, the natural map* $\mathbf{C}^{\mathrm{EqvRefl}} \longrightarrow \mathbf{C}^{\mathrm{Eqv}}$ *preserves such structure.*

Proof. Again, a direct application of Proposition 5.1. For the Π_{ext}-types, a generating set of epis is given by l_0 and pl. □

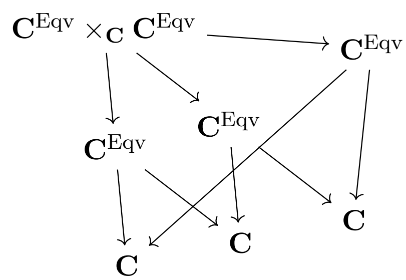
Finally, we show that $\mathbf{C}^{\mathrm{EqvRefl}}$ can be viewed as a weak map from $\mathbf{C}$ to $\mathbf{C}^{\mathrm{Eqv}}$ as promised.

Proposition 5.16. *The projection map* $\mathrm{ev}_0 : \mathbf{C}^{\mathrm{EqvRefl}} \longrightarrow \mathbf{C}$ *is a local trivial fibration.*

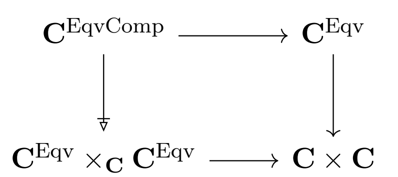
Proof. By Propositon 5.1, since the inclusion of (0) in EqvRefl is an injective simple equivalence. □

Composites of spans

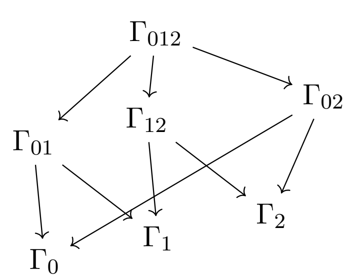
As with reflexivity, one would hope for a "composition" map on span-equivalences, of the form:



Again, however, it seems difficult to define such a map in general, so we construct it as a weak map, i.e. a left-trivial span over $\mathbf{C} \times \mathbf{C}$:



Roughly, an object of $\mathbf{C}^{\mathrm{EqvComp}}$ should consist of a pair of "input" equivalences; an "output" equivalence; and a homotopy from the composite of the input pair to the output pair. Translated entirely into span-equivalences, this becomes a diagram

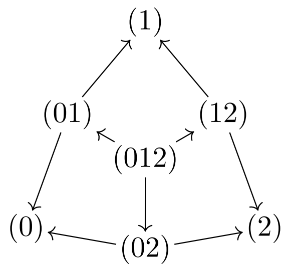


in which all maps are equivalences.

(Think of Γ_{012} as a span from $\Gamma_{01} \times_{\Gamma_1} \Gamma_{12}$ to Γ_{02} over $\Gamma_0 \times \Gamma_2$, but expressed in a way that doesn't assume existence of that pullback.)

Flattened out, the domain of the above diagram is a familiar object: the category of faces of the 2-simplex. Concretely,

Definition 5.17. EqvComp is the posetal category generated by the graph



We consider it as a homotopical inverse category, with all maps weak equivalences.

Definition 5.18. $\mathbf{C}^{\mathrm{EqvComp}}$ is the CwA of homotopical diagrams on EqvComp in $\mathbf{C}$.

Remark 5.19. In more traditional type-theoretic notation, suppose $\vec{\Gamma}$ is a homotopical diagram on EqvComp, with objects and maps denoted as e.g. $p_{0\hat{1}2} : \Gamma_{012} \longrightarrow \Gamma_{02}$. Then a type over $\vec{\Gamma}$ in $\mathbf{C}^{\mathrm{EqvComp}}$ consists of types

$$\Gamma_0 \vdash A_0 \ \mathsf{type} \qquad \Gamma_1 \vdash A_1 \ \mathsf{type} \qquad \Gamma_2 \vdash A_2 \ \mathsf{type} \qquad \Gamma_{01},\, x_0{:}l^*_{0\hat{1}}A_0,\, x_1{:}l^*_{\hat{0}1}A_1 \vdash A_{01}(x_0,x_1) \ \mathsf{type}$$
$$\Gamma_{12},\, x_1{:}l^*_{1\hat{2}}A_1,\, x_2{:}l^*_{\hat{1}2}A_2 \vdash A_{12}(x_1,x_2) \ \mathsf{type} \qquad \Gamma_{02},\, x_0{:}l^*_{0\hat{2}}A_0,\, x_2{:}l^*_{\hat{0}2}A_2 \vdash A_{02}(x_0,x_2) \ \mathsf{type}$$
$$\Gamma_{012},\, x_0{:}l^*_{0\hat{1}\hat{2}}A_0,\, x_1{:}l^*_{\hat{0}1\hat{2}}A_1,\, x_2{:}l^*_{\hat{0}\hat{1}2}A_2,\, x_{01}{:}l^*_{01\hat{2}}A_{01}(x_0,x_1),$$
$$x_{12}{:}l^*_{\hat{0}12}A_{12}(x_1,x_2),\, x_{02}{:}l^*_{0\hat{1}2}A_{02}(x_0,x_2) \vdash A_{012}(x_0,x_1,x_2,x_{01},x_{12},x_{02}) \ \mathsf{type}$$

such that the following context extensions are all contractible:

$$\Gamma_{01},\, x_0{:}l^*_{0\hat{1}}A_0 \vdash x_1{:}l^*_{\hat{0}1}A_1,\, x_{01}{:}A_{01}(x_0,x_1) \ \mathsf{cxt}$$
$$\Gamma_{01},\, x_1{:}l^*_{\hat{0}1}A_1 \vdash x_0{:}l^*_{0\hat{1}}A_0,\, x_{01}{:}A_{01}(x_0,x_1) \ \mathsf{cxt}$$
$$\Gamma_{12},\, x_1{:}l^*_{1\hat{2}}A_1 \vdash x_2{:}l^*_{\hat{1}2}A_2,\, x_{12}{:}A_{12}(x_1,x_2) \ \mathsf{cxt}$$
$$\Gamma_{12},\, x_2{:}l^*_{\hat{1}2}A_2 \vdash x_1{:}l^*_{1\hat{2}}A_1,\, x_{12}{:}A_{12}(x_1,x_2) \ \mathsf{cxt}$$
$$\Gamma_{02},\, x_0{:}l^*_{0\hat{2}}A_0 \vdash x_2{:}l^*_{\hat{0}2}A_2,\, x_{02}{:}A_{02}(x_0,x_2) \ \mathsf{cxt}$$
$$\Gamma_{012},\, x_0{:}l^*_{0\hat{1}\hat{2}}A_0,\, x_1{:}l^*_{\hat{0}1\hat{2}}A_1,\, x_2{:}l^*_{\hat{0}\hat{1}2}A_2,\, x_{01}{:}l^*_{01\hat{2}}A_{01}(x_0,x_1),\, x_{12}{:}l^*_{\hat{0}12}A_{12}(x_1,x_2)$$
$$\vdash x_{02}{:}l^*_{0\hat{1}2}A_{02}(x_0,x_2),\ x_{012}{:}A_{012}(x_0,x_1,x_2,x_{01},x_{12},x_{02}) \ \mathsf{cxt}$$

(Again, these form a minimal subclass ensuring that all maps in the resulting context extension are equivalences.)

Proposition 5.20.

- $\mathbf{C}^{\mathrm{EqvComp}}$ *carries* Id*-types;*
- *if* $\mathbf{C}$ *additionally carries* Σ*- types (resp. unit types) then so does* $\mathbf{C}^{\mathrm{EqvComp}}$*;*
- *if* $\mathbf{C}$ *has* Π*-types and functional extensionality, then so does* $\mathbf{C}^{\mathrm{Eqv}}$*;*
- *moreover, in all these cases, the natural map* $\mathbf{C}^{\mathrm{EqvComp}} \longrightarrow \mathbf{C}^{\mathrm{Eqv}}$ *preserves such structure.*

Proof. Once again, a direct application of Proposition 5.1. For the Π_{ext}-types, since EqvComp is posetal, all maps are epis. □

Finally, we once again must show that $\mathbf{C}^{\mathrm{EqvComp}}$ can be viewed as a weak map as intended.

Proposition 5.21. *The projection map* $\mathrm{ev}_0 : \mathbf{C}^{\mathrm{EqvComp}} \longrightarrow \mathbf{C}^{\mathrm{Eqv}} \times_{\mathbf{C}} \mathbf{C}^{\mathrm{Eqv}}$ *is a local trivial fibration.*

Proof. By Proposition 5.1, since $\mathbf{C}^{\mathrm{Eqv}} \times_{\mathbf{C}} \mathbf{C}^{\mathrm{Eqv}} \cong \mathbf{C}_h^{\mathrm{Eqv}+_1\mathrm{Eqv}}$, and the inclusion of $\mathrm{Eqv}+_1\mathrm{Eqv}$ into EqvComp is a simple equivalence. □

Remark 5.22. Astute readers may notice that the final propositions of these subsections have effectively shown:

- $\mathbf{C}^{\mathrm{Eqv}}$ forms a Reedy span-equivalence from $\mathbf{C}$ to itself;
- $\mathbf{C}^{\mathrm{Eqv}}$ together with $\mathbf{C}^{\mathrm{EqvRefl}}$ forms a trivial auto-equivalence of $\mathbf{C}$;

- $\mathbf{C}^{\mathrm{Eqv}}$ together with $\mathbf{C}^{\mathrm{EqvComp}}$ forms a commuting triangle of equivalences over $\mathbf{C}$.

The authors did not notice this until quite late in the preparation of this article.

6. The left semi model structure on contextual categories

We now have all the main ingredients prepared to deduce that the three classes of maps introduced in Section 3 form a left semi-model structure.

In this section, we first bring the span-equivalences construction back to the contextual world, and use it to define homotopy between maps of contextual categories. We then recall the definition of left semi-model structure, and show with just a little diagram chasing that we have one on our hands.

Returning to the contextual world

The CwA's $(-)^{\mathrm{Eqv}}$, $(-)^{\mathrm{EqvRefl}}$ and $(-)^{\mathrm{EqvComp}}$ of the previous section will almost never be contextual. To bring them back to the contextual setting, we take their cores.

Making liberal use of Proposition 4.13 (that core sends local (trivial) fibrations to (trivial) fibrations), together with the fact that core is a coreflection (so it preserves limits, and $\operatorname{core}\mathbf{C} \cong \mathbf{C}$ when $\mathbf{C}$ is contextual), we sum up the result:

Proposition 6.1. *For each $\mathbf{C}$ in $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$, we have diagrams as follows, all in $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$, and functorial in $\mathbf{C}$:*

$$\begin{array}{ccccc} & & \operatorname{core}\mathbf{C}^{\mathrm{Eqv}} & & \\ & \swarrow & \downarrow & \searrow & \\ \mathbf{C} & \longleftarrow & \mathbf{C}\times\mathbf{C} & \longrightarrow & \mathbf{C} \end{array} \qquad \begin{array}{ccc} \operatorname{core}\mathbf{C}^{\mathrm{EqvRefl}} & \longrightarrow & \operatorname{core}\mathbf{C}^{\mathrm{Eqv}} \\ \downarrow & & \downarrow \\ \mathbf{C} & \xrightarrow{\Delta} & \mathbf{C}\times\mathbf{C} \end{array}$$

$$\begin{array}{ccc} \operatorname{core}\mathbf{C}^{\mathrm{EqvComp}} & \xrightarrow{P_{02}} & \operatorname{core}\mathbf{C}^{\mathrm{Eqv}} \\ {\scriptstyle (P_{01},P_{12})}\downarrow & & \downarrow \\ \operatorname{core}\mathbf{C}^{\mathrm{Eqv}} \times_{\operatorname{core}\mathbf{C}} \operatorname{core}\mathbf{C}^{\mathrm{Eqv}} & \xrightarrow{(P_0,P_2)} & \mathbf{C}\times\mathbf{C} \end{array}$$

□

For readability, for the remainder of this section, we will omit the "core" and write just $\mathbf{C}^{\mathrm{Eqv}}$ and so on, since we have no further need of the CwA versions.

The right homotopy relation

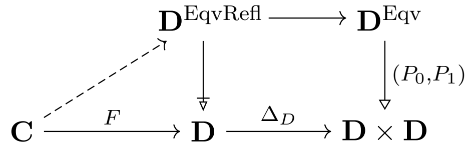
Using $(-)^{\mathrm{Eqv}}$ as a path-object construction, we can define a notion of right homotopy between maps in $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$, which will be well-behaved under cofibrant domains.

Definition 6.2. Say $F_0, F_1 : \mathbf{C} \longrightarrow \mathbf{D}$ in $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$ are **right homotopic** ($F_0 \sim_r F_1$) if they factor jointly through $\mathbf{D}^{\mathrm{Eqv}}$:

$$\begin{array}{ccc} & \overset{H}{\nearrow} & \mathbf{D}^{\mathrm{Eqv}} \\ & & \downarrow{\scriptstyle (P_0,P_1)} \\ \mathbf{C} & \xrightarrow{(F_0,F_1)} & \mathbf{D}\times\mathbf{D} \end{array}$$

Proposition 6.3. *When $\mathbf{C}$ is cofibrant, right homotopy is an equivalence relation on $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}(\mathbf{C},\mathbf{D})$.*

Proof. Reflexivity: by Proposition 5.16 and cofibrancy of $\mathbf{C}$, any map $F : \mathbf{C} \longrightarrow \mathbf{D}$ lifts to a map $\mathbf{C} \longrightarrow \mathbf{D}^{\mathrm{EqvRefl}}$; composing this with the forgetful map $\mathbf{D}^{\mathrm{EqvRefl}} \longrightarrow \mathbf{D}^{\mathrm{Eqv}}$ yields a reflexivity homotopy for F:



Symmetry is immediate (and does not require the cofibrancy of $\mathbf{C}$), using the obvious symmetry automorphism on $\mathbf{D}^{\mathrm{Eqv}}$.

Transitivity is similar to reflexivity. Given $F_0, F_1, F_2 : \mathbf{C} \longrightarrow \mathbf{D}$, and homotopies H_{01}, H_{02}, we get an induced map $(H_{01}, H_{02}) : \mathbf{C} \longrightarrow \mathbf{D}^{\mathrm{Eqv}} \times_{\mathbf{D}} \mathbf{D}^{\mathrm{Eqv}}$. By Proposition 5.21 and cofibrancy of $\mathbf{C}$, we can lift this to a map $\mathbf{C} \longrightarrow \mathbf{D}^{\mathrm{EqvComp}}$; composing this with $P_{02} : \mathbf{D}^{\mathrm{EqvComp}} \longrightarrow \mathbf{D}^{\mathrm{Eqv}}$ gives a homotopy $F_0 \sim_r F_2$. □

Proposition 6.4. *Right homotopy is stable under pre- and post-composition.*

Proof. Let $H : \mathbf{C} \longrightarrow \mathbf{D}^{\mathrm{Eqv}}$ be a homotopy $F_0 \sim_r F_1$. Then for any $G : \mathbf{C}' \longrightarrow \mathbf{C}$, HG is a homotopy $F_0 G \longrightarrow F_1 G$; and similarly, for any $K : \mathbf{D} \longrightarrow \mathbf{D}'$, $K^{\mathrm{Eqv}} H$ is a homotopy $KF_0 \sim_r KF_1$. □

Proposition 6.5. *Any map right homotopic to an equivalence is an equivalence.*

Proof. By Proposition 5.10, the maps $P_i : \mathbf{D}^{\mathrm{Eqv}} \longrightarrow \mathbf{D}$ are equivalences. So by 2-out-of-3, if $H : \mathbf{C} \longrightarrow \mathbf{D}^{\mathrm{Eqv}}$ is a homotopy $F_0 \sim_r F_1$, we have that H is an equivalence if and only if each/either $F_i = P_i H$ is one. □

Putting it all together: the semi model structure

Finally, we show that the classes of maps on $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$ fit together to form a *left semi-model structure.*[4]

Roughly, this means three classes of maps as in a model structure, except that the $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ factorization system only works for maps with cofibrant domains.

Definition 6.6 (cf. [Spi01, Def. 1(I)])**.** A **left semi-model structure** on a bicomplete category $\mathcal{E}$ consists of three classes of maps: $\mathcal{W}$, $\mathcal{F}$, $\mathcal{C}$, subject to the axioms:

1. all three classes are closed under retracts; $\mathcal{W}$ satisfies the 2-out-of-3 property; and fibrations and trivial fibrations are preserved under pullback;
2. cofibrations have the left lefting property with respect to trivial fibrations; and trivial cofibrations with cofibrant source have the left lifting property with respect to fibrations;
3. every map can be functorially factored into a cofibration followed by a trivial fibration; every map with cofibrant source can also be functorially factored into a trivial cofibration followed by a fibration.

(Left semi-model categories first appeared in Hovey [Hov98, Thm. 3.3], and were further developed by Spitzweck [Spi01, Def. 1] and Barwick [Bar10, Def. 1.4].)

In practice, one usually has just a little more structure:

Lemma 6.7. *Suppose $\mathcal{E}$ is a bicomplete category, equipped with*

- *a class of maps $\mathcal{W}$, including all identities, and closed under 2-out-of-6 and retracts;*
- *two weak factorization systems $(\mathcal{A}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{TF})$;*
- *such that $\mathcal{TF} = \mathcal{F} \cap \mathcal{W}$, and*
- *when $A \in \mathcal{E}$ is cofibrant (i.e. the map $0 \longrightarrow A$ is in $\mathcal{C}$), a map $i : A \longrightarrow B$ is in $\mathcal{A}$ if and only if it is in $\mathcal{C} \cap \mathcal{W}$.*

Then the classes $(\mathcal{W}, \mathcal{C}, \mathcal{F})$ form a left semi-model structure on $\mathcal{E}$. □

[4] We do not know whether it also forms a full model structure; we have no specific obstruction or counterexample.

For the remainder of this section, we fix the ambient category as either $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma}$ or $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma,\Pi_{\mathsf{ext}}}$ (the two cases are exactly parallel), and work to establish the hypotheses of Lemma 6.7.

Proposition 6.8. *A map in* $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$ *is a trivial fibration if and only if it is both a fibration and a weak equivalence.*

Proof. It is clear that any trivial fibration is both a weak equivalence and a fibration.

For the converse, suppose $F : \mathbf{C} \longrightarrow \mathbf{D}$ is a weak equivalence and a fibration.

Given a context Γ in $\mathbf{C}$ and type A over $F\Gamma$ in $\mathbf{D}$, we may find (since F is a weak equivalence) some type A' over Γ in $\mathbf{C}$, together with an equivalence $w : F(A') \simeq A$ over Γ in $\mathbf{D}$. Choose left and right quasi-inverses for w. Since F is a fibration, we may now lift A and w together to $\mathbf{C}$. In particular, we have succeeded in lifting A on the nose, as required.

Strict lifting of terms is entirely analogous: first lift the term up to equivavalence (since $F \in \mathcal{W}$), and then use that equivalence to lift the original term on the nose (by $F \in \mathcal{F}$). ☐

Proposition 6.9. *Let* $\mathbf{C}$ *be cofibrant in* $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$*. Then a map* $F : \mathbf{C} \longrightarrow \mathbf{D}$ *in* $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathsf{ext}})}$ *is anodyne precisely if it is both a weak equivalence and a cofibration.*

Proof. $\mathcal{A} \subseteq \mathcal{C}$: this does not require the cofibrant domain assumption. We noted above that $\mathcal{TF} \subseteq \mathcal{F}$, so dually, $\mathcal{A} \subseteq \mathcal{C}$.

$\mathcal{A} \subseteq \mathcal{W}$: suppose $j : \mathbf{A} \longrightarrow \mathbf{B}$ is anodyne, with $\mathbf{A}$ cofibrant. By fibrancy of $\mathbf{A}$ (Proposition 3.13), we can take a left inverse r for j:

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{1_A} & \mathbf{A} \\ {\scriptstyle j}\downarrow & \overset{r}{\nearrow} & \downarrow \\ \mathbf{B} & \longrightarrow & 1 \end{array}$$

But then r is also a homotopy right inverse for j, by filling the square

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{c_j} & \mathbf{B}^{\mathrm{Eqv}} \\ {\scriptstyle j}\downarrow & \overset{H}{\nearrow} & \downarrow \\ \mathbf{B} & \xrightarrow{(1_B, jr)} & \mathbf{B} \times \mathbf{B} \end{array}$$

where c_j is a reflexivity homotopy on j, supplied by Proposition 6.3 since $\mathbf{A}$ is cofibrant.

So $rj = 1_A$, and $jr \sim_r 1_B$; so by 2-out-of-6 and Proposition 6.5, j is an equivalence.

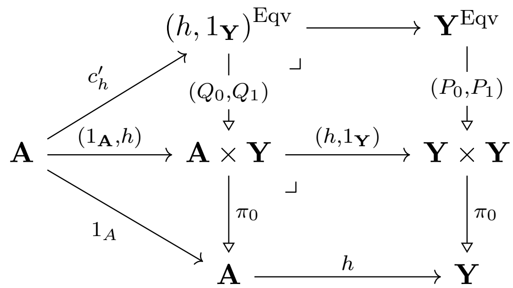
$\mathcal{W} \cap \mathcal{C} \subseteq \mathcal{A}$: suppose $j : \mathbf{A} \longrightarrow \mathbf{B}$ is in $\mathcal{W}$ and $\mathcal{C}$, with $\mathbf{A}$ cofibrant.

We want to show that j is orthogonal to all fibrations. It is enough to show this for fibrations over $\mathbf{B}$, since any other lifting problem can first be pulled back to $\mathbf{B}$. So assume $p : \mathbf{Y} \longrightarrow \mathbf{B}$ is some fibration, with a map $h : \mathbf{A} \longrightarrow \mathbf{Y}$ over $\mathbf{B}$; we want to fill the square

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{h} & \mathbf{Y} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbf{B} & \xrightarrow{1_B} & \mathbf{B}. \end{array}$$

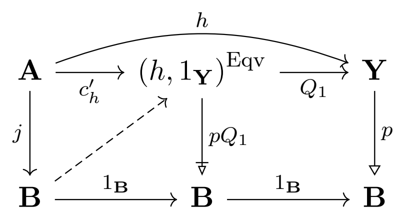
Proposition 6.3 gives a reflexivity homotopy $c_h : \mathbf{A} \longrightarrow \mathbf{Y}^{\mathrm{Eqv}}$ for h. Write $(h, 1_{\mathbf{Y}})^{\mathrm{Eqv}}$ for the pullback $(h, 1_{\mathbf{Y}})^* \mathbf{Y}^{\mathrm{Eqv}}$, with projection maps $(Q_0, Q_1) : (h, 1_{\mathbf{Y}})^{\mathrm{Eqv}} \longrightarrow \mathbf{A} \times \mathbf{Y}$. Then c_h

factors through $(h, 1_{\mathbf{Y}})^{\mathrm{Eqv}}$ by a map c'_h:



Now $Q_0 : (h, 1_{\mathbf{Y}})^{\mathrm{Eqv}} \longrightarrow \mathbf{A}$ is a pullback of $P_0 : \mathbf{Y}^{\mathrm{Eqv}} \longrightarrow \mathbf{Y}$; so by Proposition 5.10, it is a trivial fibration. So by 2-of-3, c'_h is an equivalence, since $Q_0 c'_h = 1_{\mathbf{A}}$, and by 2-of-3 again, so is pQ_1, since $pQ_1 c'_h = ph = j$.

But pQ_1 is also a fibration (as a composite of two fibrations); so by Proposition 6.8, it is a trivial fibration. So (since j is a cofibration) we can extend c'_h along j, filling the left-hand square below; composing the resulting filler with Q_1 then solves the original lifting problem.



□

This completes the main result:

Theorem 6.10. *On each of* $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma}$ *and* $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma,\Pi_{\mathrm{ext}}}$*, the classes* $\mathcal{W}$*,* $\mathcal{F}$*,* $\mathcal{C}$ *of Section 3 form a left semi-model structure.*

Proof. Propositions 6.8 and 6.9 supply the hypotheses of Proposition 6.7. □

Corollary 6.11. *The cofibrant objects of* $\mathrm{CxlCat}_{\mathsf{Id},1,\Sigma(,\Pi_{\mathrm{ext}})}$ *form a cofibration category.*

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