

Introduction to the Joyal model structure

Talk by Dan Christensen, notes by Luis Scoccola

Reading seminar on Higher Category Theory
University of Western Ontario, Fall 2016

The goal of this introduction is to discuss some aspects of the Joyal model structure on \mathbf{sSet} and to compare it to the Quillen model structure. In particular we want to understand weak categorical equivalences, which are the weak equivalences in the Joyal model structure. References for the topics in these notes are:

- [Joy08] for the general theory and the 2-categorical aspects, this contains statements and definitions.
- [Joy09] and [Rez16] also for the general theory, but these contain proofs and more details.
- [Jar15] and [DS11] for a more technical account of the construction of the Joyal model structure.

We know that every model structure is determined by cofibrations and fibrant objects, the Joyal model structure has as cofibrations all monomorphisms and as fibrant objects, quasicategories. On the other hand, in the Quillen model structure cofibrations are monomorphisms and fibrant objects are Kan complexes.

We start with some remarks that allow us abstract the notions that appear in both model structures. The idea is to use enriched category theory to simplify the simplicially enriched structure of the category of simplicial sets.

Since \mathbf{sSet} is a Cartesian closed category (defining $(Y^X)_n := \mathbf{sSet}(X \times \Delta_n, Y)$) it is enriched over itself. This means that for any two simplicial sets X, Y we can form the internal hom Y^X which represents the simplicial morphisms $X \rightarrow Y$. And moreover, that given simplicial sets X, Y, Z we have a simplicial map $Y^X \times Z^Y \rightarrow Z^X$ that represents composition.

We will need to transform the self-enrichment of \mathbf{sSet} to an enrichment over other categories, the following is a general method to do this. Given a category $\mathcal{C}_{\mathcal{V}}$ enriched over some monoidal category \mathcal{V} and a monoidal functor $\mathcal{F} : \mathcal{V} \rightarrow \mathcal{U}$ to some other monoidal category \mathcal{U} , we get an induced category $\mathcal{C}_{\mathcal{U}}$ enriched over \mathcal{U} by the following construction: Take the objects of $\mathcal{C}_{\mathcal{U}}$ to be the same objects as the objects of $\mathcal{C}_{\mathcal{V}}$. Take $\mathrm{hom}_{\mathcal{C}_{\mathcal{U}}}(a, b) := \mathcal{F}(\mathrm{hom}_{\mathcal{C}_{\mathcal{V}}}(a, b))$. The fact that \mathcal{F} is monoidal implies that we have identities, using the identities in $\mathcal{C}_{\mathcal{V}}$, and a well defined associative composition.

Using these tools we can construct three enriched categories that can be regarded as different truncations of the (simplicially enriched) category \mathbf{sSet} .

Definition 1. Consider the functor $\pi_0 : \mathbf{sSet} \rightarrow \mathbf{Set}$, that sends each simplicial set to the set of its connected components. The category \mathbf{Set} is Cartesian closed and π_0 preserves finite products, so it is monoidal. Using the construction in the previous remark we get a category (enriched over \mathbf{Set} , and thus a standard category) \mathbf{sSet}^{π_0} that has as objects simplicial sets and as maps homotopy classes of simplicial maps.

The following theorem can be used as a definition of weak equivalence for the Quillen model structure in \mathbf{sSet} . Recall that a weak equivalence is usually defined as simplicial map such that its geometric realization is a weak homotopy equivalence (between topological spaces).

Theorem 2 (Gabriel-Zisman). *A simplicial map $u : X \rightarrow Y$ is a weak equivalence (in the Quillen model structure on \mathbf{sSet}) if and only if $\pi_0(u)^* : \pi_0(Y, K) \rightarrow \pi_0(X, K)$ is a bijection for every Kan complex K .* ■

Notice that this is a completely combinatorial characterization and it will be an important fact to keep in mind when comparing the Quillen model structure with the Joyal model structure, which is our next goal.

Definition 3. Now consider the functor $\tau_1 : \mathbf{sSet} \rightarrow \mathbf{Cat}$, that we can define as the extension:

$$\begin{array}{ccc} \Delta & \longrightarrow & \mathbf{Cat} \\ y \downarrow & \nearrow \tau_1 & \\ \mathbf{sSet} & & \end{array}$$

or as the left adjoint of the usual nerve functor $\tau_1 \dashv \mathbf{N}$. The category \mathbf{Cat} is cartesian closed and τ_1 is monoidal, so using the construction in the previous remark we get a category enriched over \mathbf{Cat} , thus a 2-category, which we denote by \mathbf{sSet}^{τ_1} .

Definition 4. Finally we compose τ_1 with the functor $\mathbf{Cat} \rightarrow \mathbf{Set}$ that sends a (small) category to the set of its isomorphism classes. We call this composition τ_0 and using again the construction we get a (standard) category \mathbf{sSet}^{τ_0} .

The following is a concrete and useful construction of the functor τ_1 .

Proposition 5. *Given a simplicial set X take F to be the free category on the graph $X_0 \leftarrow X_1$ (here X_0 is the set of vertices and X_1 the set of edges, the maps are the two face maps that determine the source and target of the edges). So the objects of F are the vertices of X and the arrows of F are sequences of composable 1-simplices of X .*

Define $\tau_1(X)$ as the quotient of F by the relation induced by:

- For every $x \in X_0$, $[] \sim [\text{Id}_x]$.
- For every 2-simplex $t \in X_2$, $[s_2(t), s_0(t)] \sim [s_1(t)]$ ■

Using this construction we can define the map:

$$\begin{aligned} X &\rightarrow \mathbf{N}(\tau_1(X)) \\ \sigma &\mapsto [\sigma_{n,n-1}, \dots, \sigma_{1,0}]. \end{aligned}$$

One can check that this is in fact the unit of the adjunction $\tau_1 \dashv \mathbf{N}$.

Notice that if C is a quasicategory then in the previous construction of τ_1 every composable sequence is actually equivalent to one of length 1. Moreover one can check that also in the case C is a quasicategory $\tau_1(C)$ is equivalent to the homotopy category of C , defined using vertices as objects and simplicial homotopy classes of 1-simplices as arrows.

Now we give the basic definitions needed to describe the Joyal model structure.

Definition 6. A simplicial map is a *categorical equivalence* if it is an equivalence in \mathbf{sSet}^{τ_1} .

A simplicial map $u : A \rightarrow B$ is a *weak categorical equivalence* if $\tau_0(u)^* : \tau_0(B, C) \rightarrow \tau_0(A, C)$ is a bijection for all quasicategories C .

The following lemma follows immediately from the definition.

Lemma 7. *A simplicial map is a categorical equivalence if and only if it is an isomorphism in \mathbf{sSet}^{τ_0} .*

The following is a key property of quasicategories, the proof of this fact is not straightforward and depends on the construction of the Joyal model structure.

Proposition 8. *For any simplicial set X and any quasicategory C , the simplicial set C^X is again a quasicategory.* ■

Proposition 9. *A simplicial map $u : X \rightarrow Y$ is a weak categorical equivalence if and only if $\tau_1(u)^* : \tau_1(Y, C) \rightarrow \tau_1(X, C)$ is an equivalence for all quasicategories C .*

Proof. The *if* part is immediate, since an equivalence of categories induces a bijection on isomorphism classes.

For the *only if* notice that by Proposition 8, we know that $\tau_0(u, C^S)$ is a bijection for any simplicial set S and any quasicategory C . Now the exponential law tells us that $\tau_0(S, C^u)$ is a bijection, and so C^u is a categorical equivalence. But then $\tau_1(1, C^u)$ is an equivalence of categories, and this map is $\tau_1(u, C)$. ■

Theorem 10 (Joyal model structure). *The category \mathbf{sSet} has a model structure where:*

- *Cofibrations are monomorphisms.*
- *Weak equivalences are weak categorical equivalences. Explicitly, a weak equivalence is simplicial map $u : X \rightarrow Y$ that induces an equivalence (between categories) $\tau_1(u)^* : \tau_1(Y, C) \rightarrow \tau_1(X, C)$ for every quasicategory C .*
- *Fibrant objects are quasicategories.*

We denote this model category by \mathbf{sSet}_J . ■

While weak categorical equivalences between arbitrary simplicial sets are difficult to identify, the situation is simpler between quasicategories.

Lemma 11. *If C, D are quasicategories, then $u : C \rightarrow D$ is a weak categorical equivalence if and only if it is a categorical equivalence.* ■

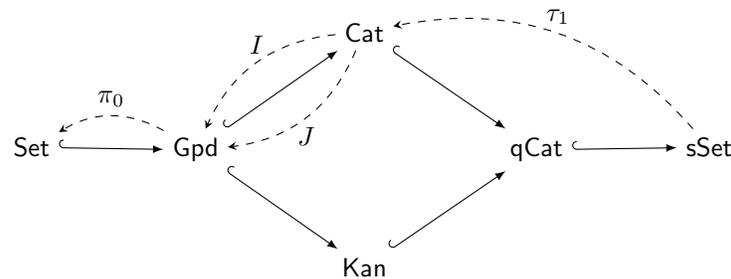
We conclude with a diagrammatic description of the difference between the Quillen and the Joyal model structures. By Theorem 2 we can describe the Quillen model structure as follows.

Theorem 12 (Quillen model structure). *The category \mathbf{sSet} has a model structure where:*

- *Cofibrations are monomorphisms.*
- *Weak equivalences are simplicial maps $u : X \rightarrow Y$ that induce bijections $\pi_0(u)^* : \pi_0(Y, K) \rightarrow \pi_0(X, K)$ for every Kan complex K .*
- *Fibrant objects are Kan complexes.*

We denote this model category by \mathbf{sSet}_Q . ■

Notice that Proposition 9 together with Lemma 11 give us an analogue of Theorem 2 for quasicategories. Then the difference between weak equivalences in the Quillen model structure and weak equivalences in the Joyal model structure can be visualized as follows:



Here dashed functors are adjoints, and more precisely a functor on top of another is left adjoint to this functor. The functor I (freely) inverts all arrows, while J restricts a category to its subcategory spanned by isomorphisms (this functor is also called “core”). Then, the π_0 that appears in Theorem 2 is the composition $\pi_0 \circ I \circ \tau_1$, while the τ_0 that defines weak equivalences in the Joyal model structure is the composition $\pi_0 \circ J \circ \tau_1$.

References

- [DS11] Daniel Dugger and David I Spivak. Mapping spaces in quasi-categories. *Algebraic & Geometric Topology*, 11(1):263–325, 2011.
- [Jar15] JF Jardine. Path categories and quasi-categories. *preprint*, 2015.
- [Joy08] André Joyal. Notes on quasi-categories. *preprint*, 2008.
- [Joy09] André Joyal. The theory of quasi-categories and its applications. *preprint*, 2009.
- [Rez16] Charles Rezk. Stuff about quasicategories. *Higher Category Theory & Quasicategories, Course notes*, 2016.