MODELS FOR $(\infty, 1)$-CATEGORIES: PART 1

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Introduction: beyond quasi-categories

Roughly speaking, a *homotopy theory* $\mathcal{H}$ consists of the following data:

(H1) a *homotopy category* $\mathrm{Ho}(\mathcal{H})$;
(H2) for objects $x, y \in \mathrm{Ho}(\mathcal{H})$, a *mapping space* (or*homotopy function complex*) $\mathrm{Map}_{\mathcal{H}}(x, y) \in \mathsf{sSet}$.

A *presentation* (or a *model*) for a homotopy theory is then any mathematical object which gives rise to a homotopy theory, e.g. quasi-categories. Many presentations of homotopy theories do *not* commonly arise as quasi-categories but, rather, in other ways, like relative categories or simplicial categories. So, even if the category QCat of quasi-categories allows a formal, well-studied treatment of homotopy theories, it does not really include many of the natural examples.

To address this issue, one can follow a general leading principle.

Principle (Homotopy Invariance)**.** To each presentation X of a homotopy theory $\mathcal{H}$, (functorially) associate a quasi-category RX presenting the same underlying homotopy theory $\mathcal{H}$.

If this is the case, we can be sure that:

- there is a way to see each presentation X of a homotopy theory $\mathcal{H}$ as a quasi-category, without losing any homotopical information;
- up to equivalence of associated homotopy theories, one is free to work in X or in the associated quasi-category, as convenience suggests.

Often one can organize all presentations of the same sort for homotopy theories (such as relative or simplicial categories) in a category HMod which is, itself, a model for a homotopy theory. In this case we would like to get a functor $R\colon \mathsf{HMod} \to \mathsf{QCat}$ inducing equivalences between the data (H1) and (H2) for HMod and QCat.

The goal of these notes is to show that, for several instances of HMod (namely, for complete Segal spaces, simplicial categories and relative categories), one can

(a) endow HMod with a model category structure;
(b) find a functor $R'\colon \mathsf{HMod} \to (\mathsf{sSet})_{\mathrm{Joyal}}$ which is the right adjoint in a Quillen equivalence.

Since the right derived functor of right Quillen functors automatically induces weak equivalences at the level of homotopy function complexes, we can take our functor R to be the right derived functor of R'.

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1. Complete Segal Spaces: the bisimplicial model

Complete Segal spaces are certain kinds of bisimplicial sets introduced in [Rez01]. They are the fibrant objects of a model category CSs which is Quillen equivalent to $(\mathsf{sSet})_{\mathrm{Joyal}}$, thus they present $(\infty,1)-$categories. This presentation has some remarkable properties.

(1) CSs is a *simplicial* model category. This is not the case for $(\mathsf{sSet})_{\mathrm{Joyal}}$[1]. Here is an easy way to see this. If the Joyal model structure was simplicial, since $\Delta[0]$ is cofibrant,
$$\mathrm{Map}_{\mathsf{sSet}}(\Delta[0],\bullet)\colon (\mathsf{sSet})_{\mathrm{Joyal}} \to (\mathsf{sSet})_{\mathrm{Quillen}}$$
would be a right Quillen functor. But then, any quasi-category X would also be a Kan complex because $\mathrm{Map}_{\mathsf{sSet}}(\Delta[0],X)\cong X$.

(2) There are sectionwise criteria to detect weak equivalences between complete Segal spaces. In fact, a map $g\colon X\to Y$ between complete Segal spaces is a weak equivalence in CSs if and only if it induces Kan-Quillen equivalences between the columns *or* Joyal equivalences between the rows of X and Y.

Following the work of [Rez01], in §1.1, we define the fundamental notion of *Segal space*, give a sketch of how it presents a homotopy theory and introduce the completeness condition. After [JT07], in §1.2, we exhibit two ways in which the model category structure CSs presenting complete Segal spaces is Quillen equivalent to the Joyal model structure on sSet presenting quasi-categories. The Appendix (§2) gathers the facts we need about (some of) the homotopy theories available for bisimplicial sets.

1.1. Segal spaces.

Segal spaces homotopically generalize categories via the *Segal condition*; up to a specified weak equivalence, the m-th column space of a Segal space X is an iterated pullback of its first column space. In this way, one recovers the fact that an m-simplex of the nerve of a category is just determined by the chain of 1-simplices connecting the vertices of the m-simplex. A Segal space has a set of *objects*, (homotopic) maps between them and composite of those. There are two natural notions of "sameness" for objects x,y of a Segal space X. Namely, one could ask that x and y are in the same path-component of X_0, or that there is a *homotopy equivalence* from x to y. Those Segal spaces for which these two notions coincide are called *complete Segal space.*

1.1.1. *From categories to Segal spaces.* We start with an observation about ordinary categories. Let $\mathcal{C}$ be a small category and $N\mathcal{C}\in\mathsf{sSet}$ be its nerve. For every $m\in\mathbb{N}_{\geq 1}$, we can consider the iterated pullback
$$N(\mathcal{C})_1\times_{N(\mathcal{C})_0}\cdots\times_{N(\mathcal{C})_0}N(\mathcal{C})_1 := \lim(N\mathcal{C}_1\xrightarrow{d_0}N\mathcal{C}_0\xleftarrow{d_1}N\mathcal{C}_1\xrightarrow{d_0}\cdots\xrightarrow{d_0}N\mathcal{C}_0\xleftarrow{d_1}N\mathcal{C}_1)$$
with m copies of $N\mathcal{C}_1$ on both sides of the definition. There is a map
$$N(\mathcal{C})_m\to N(\mathcal{C})_1\times_{N(\mathcal{C})_0}N(\mathcal{C})_1\times_{N(\mathcal{C})_0}\cdots\times_{N(\mathcal{C})_0}N(\mathcal{C})_1, \tag{1}$$
sending a string of m composable arrows in $\mathcal{C}$ to the $m-$tuple of those arrows. This map is clearly an isomorphism. If we look at $N\mathcal{C}$ as a discrete bisimplicial set, the map in (1) becomes a Kan-Quillen equivalence of (discrete) simplicial sets. One could try to define a similar map for a general bisimplicial set and ask whether it is a Kan-Quillen equivalence. Since some care is needed in doing this, the construction we will present here only gives the homotopical generalization of (1) for vertically fibrant bisimplicial sets.

For every $m\in\mathbb{N}_{\geq 1}$ and every $0\leq i\leq m-1$, consider the following map in Δ:
$$\alpha^i\colon [1]\to[m],\qquad 0\mapsto i,\ 1\mapsto i+1. \tag{2}$$

[1] At least with respect to the self-enrichment of sSet as a cartesian closed category.

Each α^i can be seen as a map $\alpha^i\colon \Delta[1] \to \Delta[m]$ of discrete bisimplicial sets. We set

$$I(m) := \bigcup_{i=0}^{m-1} \mathrm{Im}(\alpha^i) \subseteq \Delta[m]$$

and let

$$\varphi^m\colon I(m) \to \Delta[m] \tag{3}$$

be the inclusion. For every $m \in \mathbb{N}_{\geq 1}$, there is an isomorphism

$$\mathrm{Map}_{\mathsf{s^2Set}}(I(m), X) \cong X_1 \times_{X_0} \cdots \times_{X_0} X_1, \tag{4}$$

natural in $X \in \mathsf{s^2Set}$, where the right-hand side is

$$\lim(X_1 \overset{d_0}{\to} X_0 \overset{d_1}{\leftarrow} X_1 \overset{d_0}{\to} X_0 \overset{d_1}{\leftarrow} X_1 \cdots \overset{d_0}{\to} X_0 \overset{d_1}{\leftarrow} X_1). \tag{5}$$

Under the isomorphisms $\mathrm{Map}_{\mathsf{s^2Set}}(\Delta[m],\ X) \cong X_m$ and (4), we can then define

$$\varphi_m = \mathrm{Map}_{\mathsf{s^2Set}}(\varphi^m,\ X)\colon X_m \to X_1 \times_{X_0} \cdots \times_{X_0} X_1. \tag{6}$$

Definition 1.1. A bisimplicial set X is a *Segal space* if it is vertically fibrant and, for every $m \geq 1$, the $m-$th Segal map φ_m is a Kan-Quillen equivalence of simplicial sets.

Remark 1.2.

(a) The map φ_m can also be described as

$$\varphi^m \backslash X\colon \Delta[m]\backslash X \to I(m)\backslash X$$

(see Appendix, (28)), since $I(m)\backslash X \cong X_1 \times_{X_0} \cdots \times_{X_0} X_1$.

(b) Since φ^m is a cofibration of bisimplicial sets, if X is vertically fibrant, then φ_m is a Kan fibration. Hence, if X is a Segal space, φ_m is a trivial Kan fibration.

(c) If X is vertically fibrant, the degeneracies $d_0, d_1\colon X_1 \to X_0$ are Kan fibrations (see Lemma 2.14), so the limit $X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is actually a homotopy limit.

Remark 1.3. The Segal condition is so called because it firstly appeared in [Seg74] as a condition to understand when a space (simplicial set) has the homotopy type of a loop space (*delooping problem*). Segal's result says that, if X is a Segal space such that X_0 is contractible and X_1 is connected (although this last condition can be relaxed), then X_1 has the homotopy type of $\Omega|X|$, where $|X| = \mathrm{diag}(X)$ is the diagonal of the bisimplicial set X.

Note that Definition 1.1 says that a bisimplicial set X is a Segal space if and only if it is an S-local bisimplicial set, where S is the set of all maps φ^m in (3) (see §2.5). We can then give the following

Definition 1.4. The *Segal space model category* is the model category Ss obtained as the left Bousfield localization of $(\mathsf{s^2Set})_v$ at the set S of maps $\varphi^m\colon I(m) \to \Delta[m]$ of discrete bisimplicial sets.

Remark 1.5. By definition (and by Theorem 2.23), the fibrant objects of Ss are precisely the Segal spaces, a vertical weak equivalence of bisimplicial sets is a weak equivalence in Ss and a map between Segal spaces is a weak equivalence (resp. a fibration) in Ss if and only if it is a vertical weak equivalence (resp. a vertical fibration).

Theorem 2.23 ensures that Ss is a simplicial, left proper and combinatorial model category. It has a further important property:

Proposition 1.6 ([Rez01], Thm 7.1)**.** *Ss is a cartesian closed model category. In particular, for every Segal space X and every bisimplicial set Y, the internal hom X^Y is a Segal space.*

1.1.2. *The main example.* Let $(\mathcal{C}, \mathcal{W})$ be a small relative category (see **??** below). For $n \in \mathbb{N}$, the functor category $\mathcal{C}^{[n]}$ inherits a natural structure of relative category with weak equivalences given by the natural transformations which are sectionwise weak equivalences in $\mathcal{C}$. We denote the resulting relative category by $(\mathcal{C}^{[n]}, \mathcal{W}(\mathcal{C}^{[n]}))$.

Definition 1.7. Let $(\mathcal{C}, \mathcal{W})$ be a small relative category. The *classification diagram* of $(\mathcal{C}, \mathcal{W})$ is the bisimplicial set $N_{\mathrm{Rzk}}(\mathcal{C}, \mathcal{W})$ defined by

$$N_{\mathrm{Rzk}}(\mathcal{C}, \mathcal{W})_m := N(\mathcal{W}(\mathcal{C}^{[m]})), \qquad m \in \mathbb{N}$$

The action of N_{Rzk} on maps $f\colon [m] \to [k]$ in the simplex category is obtained by functoriality of N, since f induces a map $\mathcal{W}(\mathcal{C}^{[k]}) \to \mathcal{W}(\mathcal{C}^{[m]})$.

The assignment $(\mathcal{C}, \mathcal{W}) \mapsto N_{\mathrm{Rzk}}(\mathcal{C}, \mathcal{W})$ extends to a functor

$$N_{\mathrm{Rzk}}\colon \mathsf{RelCat} \to \mathsf{s^2Set}, \qquad (\mathcal{C}, \mathcal{W}) \mapsto N_{\mathrm{Rzk}}(\mathcal{C}, \mathcal{W}). \tag{7}$$

We will talk more extensively about this functor later on. As for now, note that the $(m, n)-$bisimplices of $N_{\mathrm{Rzk}}(\mathcal{C}, \mathcal{W})$ look like this

$$\begin{array}{ccccccc}
A_{0,n} & \longrightarrow & A_{1,n} & \longrightarrow & \cdots & \longrightarrow & A_{m,n} \\
\sim\uparrow & & \sim\uparrow & & & & \sim\uparrow \\
\vdots & & \vdots & & & & \vdots \\
\sim\uparrow & & \sim\uparrow & & & & \sim\uparrow \\
A_{0,1} & \longrightarrow & A_{1,1} & \longrightarrow & \cdots & \longrightarrow & A_{m,1} \\
\sim\uparrow & & \sim\uparrow & & & & \sim\uparrow \\
A_{0,0} & \longrightarrow & A_{1,0} & \longrightarrow & \cdots & \longrightarrow & A_{m,0}
\end{array}$$

where all the $A'_{i,j}$s are objects of $\mathcal{C}$ and all the vertical maps are in $\mathcal{W}$.

Given a small category $\mathcal{C}$ we can regard it as a relative category by taking the subcategory of weak equivalences to be the *core* of $\mathcal{C}$, i.e. the maximal subgroupoid of $\mathcal{C}$, consisting of all objects of $\mathcal{C}$ and all isomorphisms among them. We denote the core of $\mathcal{C}$ by $\mathrm{core}(\mathcal{C})$.

Definition 1.8. Let $\mathcal{C}$ be a small category. The *classifying diagram* of $\mathcal{C}$ (or the *Rezk nerve* of $\mathcal{C}$ or the *bisimplicial nerve* of $\mathcal{C}$) is

$$N_{\mathrm{Rzk}}(\mathcal{C}) := N_{\mathrm{Rzk}}(\mathcal{C}, \mathrm{core}(\mathcal{C})).$$

Remark 1.9. For every $\mathcal{C} \in \mathsf{Cat}$ and every $m \in \mathbb{N}$, $N_{\mathrm{Rzk}}(\mathcal{C})_m = N(\mathrm{core}(\mathcal{C}^{[m]}))$ and $N_{\mathrm{Rzk}}(\mathcal{C})_{\bullet,0} = N\mathcal{C}$ (see Convention 2.2).

For every $n \in \mathbb{N}$, let $E[n]$ be the (nerve of the) groupoid having $n+1$ distinct objects and precisely one isomorphism between any two of them:

$$E[n]: \quad 0 \overset{\cong}{\rightleftarrows} 1 \overset{\cong}{\rightleftarrows} 2 \overset{\cong}{\rightleftarrows} \cdots \overset{\cong}{\rightleftarrows} n \tag{8}$$

In other words, $E[n]$ is the fundamental groupoid of $\Delta[n]$. Let $[n] \to E[n]$ be the inclusion identifying $[n]$ with the top chain of arrows in $E[n]$. The definition of N_{Rzk} gives immediately the following

Lemma 1.10. *There is an isomorphism*

$$(N_{\mathrm{Rzk}}(\mathcal{C}))_{m,n} \cong \mathsf{Cat}([m] \times E[n], \mathcal{C}), \tag{9}$$

natural in $[m], [n] \in \Delta$ *and* $\mathcal{C} \in \mathsf{Cat}$.

Proposition 1.11 ([Rez01], Lemma 3.9)**.** *For every* $\mathcal{C} \in \mathsf{Cat}$*, each* $m-th$ *Segal map for* $N_{Rzk}(\mathcal{C})$ *is an isomorphism and* $N_{\mathrm{Rzk}}(\mathcal{C})$ *is a Segal space.*

Taking the ordinary nerve of a functor which is not an equivalence of categories can still produce a Kan-Quillen equivalence. This can not happen with N_{Rzk}.

Theorem 1.12. *The classifying diagram functor*

$$N_{\mathrm{Rzk}} \colon \mathsf{Cat} \to \mathsf{s^2Set}$$

is a fully faithful, cartesian closed functor. Furthermore, a map $f \colon \mathcal{C} \longrightarrow \mathcal{D}$ *is an equivalence of categories if and only if* $N_{\mathrm{Rzk}}(f)$ *is a vertical equivalence of bisimplicial sets.*

Proof. A map $N_{\mathrm{Rzk}}(\mathcal{C}) \to N_{\mathrm{Rzk}}(\mathcal{D})$ is completely determined by its actions on the $0-$th and on the first columns. Fully faithfulness of N_{Rzk} follows. N_{Rzk} preserves products by (9). Now,

$$N_{\mathrm{Rzk}}(\mathcal{D}^{\mathcal{C}})_{m,n} \cong \mathsf{Cat}([m] \times E[n], \mathcal{D}^{\mathcal{C}}) \cong \mathsf{Cat}([m] \times \mathcal{C}, \mathcal{D}^{E[n]})$$

and

$$(N_{\mathrm{Rzk}}(\mathcal{D})^{N_{\mathrm{Rzk}}(\mathcal{C})})_{m,n} \cong \mathsf{s^2Set}(\Delta[m] \times c_h(\Delta[n]), N_{\mathrm{Rzk}}(\mathcal{D})^{N_{\mathrm{Rzk}}(\mathcal{C})})$$

We have natural isomorphisms

$$\mathsf{s^2Set}(\Delta[m] \times c_h(\Delta[n]), N_{\mathrm{Rzk}}(\mathcal{D})^{N_{\mathrm{Rzk}}(\mathcal{C})}) \cong \mathsf{s^2Set}(\Delta[m] \times N_{\mathrm{Rzk}}(\mathcal{C}), N_{\mathrm{Rzk}}(\mathcal{D})^{c_h(\Delta[n])}) \cong$$

$$\cong \mathsf{s^2Set}(N_{\mathrm{Rzk}}([m] \times \mathcal{C}), N_{\mathrm{Rzk}}(\mathcal{D}^{E[n]})).$$

The last isomorphism follows because $N_{\mathrm{Rzk}}([m] \times \mathcal{C}) \cong N_{\mathrm{Rzk}}([m]) \times N_{\mathrm{Rzk}}(\mathcal{C}) \cong \Delta[m] \times N_{\mathrm{Rzk}}(\mathcal{C})$ and, for any category $\mathcal{C}$ (hence also for $\mathcal{C} = \mathcal{D}^{[m]}$), $\mathrm{core}(\mathcal{C}^{E[n]}) \cong \mathrm{core}(\mathcal{C})^{E[n]} \cong \mathrm{core}(\mathcal{C})^{[n]}$, so that $N(\mathrm{core}(\mathcal{C}^{E[n]})) \cong N(\mathrm{core}(\mathcal{C}))^{\Delta[n]}$. Since N_{Rzk} is fully faithful, we have

$$\mathsf{Cat}([m] \times \mathcal{C}, \mathcal{D}^{E[n]}) \cong \mathsf{s^2Set}(N_{\mathrm{Rzk}}([m] \times \mathcal{C}), N_{\mathrm{Rzk}}(\mathcal{D}^{E[n]})).$$

Thus, the canonical map $N_{\mathrm{Rzk}}(\mathcal{D}^{\mathcal{C}}) \to N_{\mathrm{Rzk}}(\mathcal{D})^{N_{\mathrm{Rzk}}(\mathcal{C})}$ is an isomorphism.
For the last claim, naturally isomorphic functors from $\mathcal{C}$ to $\mathcal{D}$ produce simplicially homotopic maps from $N_{\mathrm{Rzk}}(\mathcal{C})$ to $N_{\mathrm{Rzk}}(\mathcal{D})$, because two isomorphic functors give rise to a map $\mathcal{C} \times E[1] \to \mathcal{D}$ in Cat and $N_{\mathrm{Rzk}}(\mathcal{C}^{E[1]}) \cong N_{\mathrm{Rzk}}(\mathcal{C})^{c_h(\Delta[1])}$. Therefore, if $f \colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, then $N_{\mathrm{Rzk}}(f)$ is a vertical equivalence of bisimplicial sets. On the other hand, if $N_{\mathrm{Rzk}}(f)$ is a vertical equivalence, since both $N_{\mathrm{Rzk}}(\mathcal{C})$ and $N_{\mathrm{Rzk}}(\mathcal{D})$ are fibrant-cofibrant objects in $(\mathsf{s^2Set})_v$, it is also a simplicial homotopy equivalence. A simplicial homotopy inverse for f is a 0-simplex g of $N_{\mathrm{Rzk}}(\mathcal{C}^{\mathcal{D}})_0$, whereas the simplicial homotopies witnessing this fact are 1-simplices of $N_{\mathrm{Rzk}}(\mathcal{C}^{\mathcal{D}})_0$ and $N_{\mathrm{Rzk}}(\mathcal{D}^{\mathcal{C}})_0$. By the above, these correspond exactly to a functor $g \colon \mathcal{D} \to \mathcal{C}$ and to natural isomorphisms $fg \cong \mathrm{id}_{\mathcal{D}}$ and $gf \cong \mathrm{id}_{\mathcal{C}}$. □

1.1.3. *Homotopy Theory in a Segal space.* Segal spaces are presentations of homotopy theories in the sense of the Introduction.

Definition 1.13. Let X be a Segal space.

(1) The *set of objects* of X is the set $\mathrm{Ob}(X) := X_{0,0}$.

(2) For every $x, y \in \mathrm{Ob}(X)$, the *mapping space* between x and y is the Kan complex $\mathrm{map}_X(x,y) = \mathrm{map}(x,y)$ fitting in the pullback square

$$\begin{array}{ccc} \mathrm{map}(x,y) & \longrightarrow & X_1 \\ \downarrow & & \downarrow{\scriptstyle (d_1,d_0)} \\ \Delta[0] & \xrightarrow[(x,y)]{} & X_0 \times X_0 \end{array}$$

The 0-th simplices of $\mathrm{map}(x,y)$ are called *maps* (or *morphisms*) from x to y in X and denoted as $x \to y$.

(3) For every $x \in \mathrm{Ob}(X)$, $s_0(x) \in \mathrm{map}(x,x)$ is called the *identity map* on x and denoted by id_x.

(4) Two maps $f, g\colon x \to y$ in X are *homotopic* if they belong to the same path-component of $\mathrm{map}(x,y)$. We write $f \sim g$ to indicate that f and g are homotopic maps in X.

Remark 1.14. $\mathrm{map}(x,y)$ is both the fiber and the homotopy fiber of (d_1,d_0) over $(x,y) \in X_0 \times X_0$, because (d_1,d_0) is a Kan fibration. In particular, if (x,y) and (x',y') are in the same path-component of $W_0 \times W_0$, then $\mathrm{map}(x,y)$ and $\mathrm{map}(x',y')$ are weakly equivalent Kan complexes.

In order to define composition of maps in a Segal space X, consider, for $m \geq 1$ and $0 \leq i \leq m$, the morphisms

$$\beta^i\colon [0] \to [m], \qquad 0 \mapsto i$$

and let $\beta_i = X(\beta^i)\colon X_m \to X_1$. We get an induced map

$$(\beta_0, \dots, \beta_m)\colon X_m \to X_0^{m+1}.$$

We can think to this map as the one which associates to each m-simplex of X the ordered $(m+1)$-tuple of its vertices. We denote by $\mathrm{map}(x_0, x_1, \dots, x_m)$ the fiber of $(\beta_0, \dots, \beta_m)$ over $(x_0, x_1, \dots, x_m) \in \mathrm{Ob}(X)^{m+1}$. There is a commutative diagram of maps over X_0^{m+1}

$$\begin{array}{ccc} X_m & \xrightarrow{\varphi_m} & X_1 \times_{X_0} \times \cdots \times_{X_0} X_1 \\ & {\scriptstyle (\beta_0,\dots,\beta_m)} \searrow \quad \swarrow & \\ & X_0^{m+1} & \end{array}$$

Since the Segal map φ_m is a trivial fibration, there is an induced trivial Kan fibration between fibers

$$\varphi_{x_0,x_1,\dots,x_m}\colon \mathrm{map}(x_0, x_1, \dots, x_m) \xrightarrow{\sim} \mathrm{map}(x_{m-1}, x_m) \times \cdots \times \mathrm{map}(x_0, x_1)$$

We will abuse of notation and again call such a map φ_m.

Definition 1.15. Let X be a Segal space and $f \in \mathrm{map}(x,y)$, $g \in \mathrm{map}(y,z)$ be two maps in X. A *composite* of f and g is any map $x \to z$ of the form $d_1(\sigma)$, for $\sigma \in \mathrm{map}(x,y,z)$ such that $\varphi_2(\sigma) = (g,f)$.

Because φ_m is a trivial fibration, every two composites of f and g as above are homotopic, so we will denote any such composite by $g \circ f$. Composition is associative and unital up to homotopy.

Proposition 1.16 ([Rez01], Prop 5.4)**.** *Let $f\colon w \to x$, $g\colon x \to y$ and $h\colon y \to z$ be maps in a Segal space X. Then*

(i) $(h \circ g) \circ f \sim h \circ (g \circ f)$;

(ii) $f \circ \mathrm{id}_w \sim f \sim \mathrm{id}_x \circ f$.

For $f\colon x \to y$ in a Segal space, denote by $[f]$ its class in $\pi_0(\mathrm{map}(x,y))$.

Definition 1.17. Let X be a Segal space.

(1) The *homotopy category* of X is the category $\mathrm{Ho}(X)$ such that:
- $\mathrm{Ob}(\mathrm{Ho}(X)) := \mathrm{Ob}(X)$;
- for all $x, y \in \mathrm{Ob}(\mathrm{Ho}(X))$, $\mathrm{Ho}(X)(x,y) := \pi_0(\mathrm{map}(x,y))$;
- for all $f : x \to y$ and $g : y \to z$ in X, $[g] \circ [f] := [g \circ f]$.

(2) A map $f : x \to y$ is a *homotopy equivalence* in X if $[f]$ is invertible in $\mathrm{Ho}(X)$.

The following result shows that "being a homotopy equivalence" is homotopically invariant in a rather strong sense.

Proposition 1.18 ([Rez01], Lemma 5.8)**.** *Let $f : x \to y$ be a homotopy equivalence in a Segal space X. Then all the 0-simplices of X_1 which belong to the same path-component of f in X_1 are themselves homotopy equivalences in X.*

In particular, since, for every $x \in \mathrm{Ob}(X)$, id_x is a homotopy equivalence, the path-components of the identity maps in X_1 are all made of homotopy equivalences.

1.1.4. *The completeness condition.* Let X be a Segal space. We denote by X_{hoequiv} the subsimplicial set of X_1 generated by those path-components of X_1 containing homotopy equivalences. Since, for all $x \in \mathrm{Ob}(X)$, $\mathrm{id}_x = s_0(x)$ is a homotopy equivalence, we get

$$s_0 : X_0 \to X_{\mathrm{hoequiv}} \tag{10}$$

Definition 1.19. A bisimplicial set X is called a *complete Segal space* if it is a Segal space and the map (10) is a Kan-Quillen equivalence of simplicial sets.

So, in a complete Segal space, looking at the path-components of the identity maps exhausts all homotopy equivalences.

Our main examples of Segal spaces are, in fact, complete:

Proposition 1.20 ([Rez01], Prop 6.1)**.** *The classifying diagram $N_{\mathrm{Rzk}}(\mathcal{C})$ of a small category $\mathcal{C}$ is a complete Segal space.*

Proof. A map in the Segal space $N_{\mathrm{Rzk}}(\mathcal{C})$ is a homotopy equivalence if and only if it is an isomorphism in $\mathcal{C}$, so that $N_{\mathrm{Rzk}}(\mathcal{C}) \cong c_h(N(\mathrm{core}(\mathcal{C}^{E[1]}))$ (see (8)). The inclusion map $\mathrm{core}(\mathcal{C}) \to \mathrm{core}(\mathcal{C}^{E[1]})$ is an equivalence of categories. Upon taking ordinary nerves, we get that (10) is a Kan-Quillen equivalence. □

We now give several characterizations of the completeness condition for a Segal space. Recall from (8) that $E[1]$ denotes the nerve of the groupoid having exactly one isomorphism $0 \to 1$. We let $t : E[1] \to \Delta[0]$ be the map into the terminal simplicial set and $u_0, u_1 : \Delta[0] \to E[1]$ be the maps picking the vertices 0 and 1 respectively. For $X \in \mathsf{s^2Set}$, we have

$$E[1] \backslash X \cong \mathrm{Map}_{\mathsf{s^2Set}}(E[1], X).$$

(See Appendix, (25) and (28)). Given a Segal space X, and $x, y \in \mathrm{Ob}(X)$, we have Kan complexes $\mathrm{hoequiv}(x,y)$ and $X_0(x,y)$ fitting into the pullback squares

$$\begin{array}{ccc} \mathrm{hoequiv}(x,y) & \longrightarrow & X_{\mathrm{hoequiv}} \\ \downarrow & & \downarrow{\scriptstyle (d_1,d_0)} \\ \Delta[0] & \xrightarrow[(x,y)]{} & X_0 \times X_0 \end{array} \qquad \begin{array}{ccc} X_0(x,y) & \longrightarrow & X_0^{\Delta[1]} \\ \downarrow & & \downarrow \\ \Delta[0] & \xrightarrow[(x,y)]{} & X_0 \times X_0 \end{array}$$

Here $\mathrm{hoequiv}(x, y)$ is the subsimplicial set of $\mathrm{map}(x, y)$ generated by those path-components that contain homotopy equivalences, whereas $X_0(x, y)$ is the simplicial set of paths in X_0 starting at x and ending at y. We are now ready to state the following

Proposition 1.21 ([Rez01], Prop 6.4)**.** *The following are equivalent for a Segal space X.*

(1) X is a complete Segal space.
(2) the map $t\backslash X\colon \Delta[0]\backslash X \to E[1]\backslash X$ is a Kan-Quillen equivalence.
(3) either $u_0\backslash X\colon E[1]\backslash X \to \Delta[0]\backslash X$ or $u_1\backslash X\colon E[1]\backslash X \to \Delta[0]\backslash X$ is a Kan-Quillen equivalences.
(4) (Univalence) For all $x, y \in \mathrm{Ob}(X)$, $\mathrm{hoequiv}(x, y)$ is naturally weakly equivalent to $X_0(x, y)$ in $(\mathsf{sSet})_{\mathrm{Quillen}}$.

Remark 1.22. We named condition (4) in the above Proposition as "univalence" because it says that, for the type of objects modelled by the ∞-groupoid X_0, the notion of homotopy equivalence is equivalent to the notion of path. So the "universe of objects" for the homotopy theory presented by X is, indeed, univalent.

By Proposition 1.21, a Segal space is complete if and only if it is an $\{u_0\colon \Delta[0] \to E[1]\}$-local bisimplicial set (see §2.5).

Definition 1.23. The *complete Segal space model category* is the model category CSs obtained as the left Bousfield localization of $(\mathsf{s^2Set})_v$ at the set S consisting of the maps φ^m of (3) and the map $u_0\colon \Delta[0] \to E[1]$ of discrete bisimplicial sets.

Remark 1.24. By definition (and by Theorem 2.23), the fibrant objects of CSs are precisely the complete Segal spaces, a vertical weak equivalence of bisimplicial sets is a weak equivalence in CSs and a map between complete Segal spaces is a weak equivalence (resp. a fibration) in CSs if and only if it is a vertical weak equivalence (resp. a vertical fibration).

As for Ss, we also get the following

Proposition 1.25 ([Rez01], Prop 7.2)**.** *CSs is a cartesian closed model category. In particular, if X is a complete Segal space and Y is any bisimplicial set, then X^Y is a complete Segal space.*

The model category CSs is defined as a left Bousfield localization of $(\mathsf{s^2Set})_v$. One could have instead considered the horizontal model category structure on $\mathsf{s^2Set}$ (see §2.4).

Proposition 1.26 ([JT07], Thm 4.5)**.** *The complete Segal space model category structure CSs is a left Bousfield localization of the horizontal model structure on $\mathsf{s^2Set}$. Furthermore, a horizontally fibrant bisimplicial set is a complete Segal space if and only if it is categorically constant (see Definition 2.17).*

In particular, every horizontal equivalence is a weak equivalence in CSs.

Remark 1.27. By the above, in order to check that a map $f\colon X \to Y$ between complete Segal spaces is a weak equivalence in CSs, it is enough to show it induces Kan-Quillen equivalence between the columns *or* Joyal equivalences between the rows. This is one of the strongpoints of complete Segal spaces, which we will use to establish Quillen equivalences between CSs and $(\mathsf{sSet})_{\mathrm{Joyal}}$.

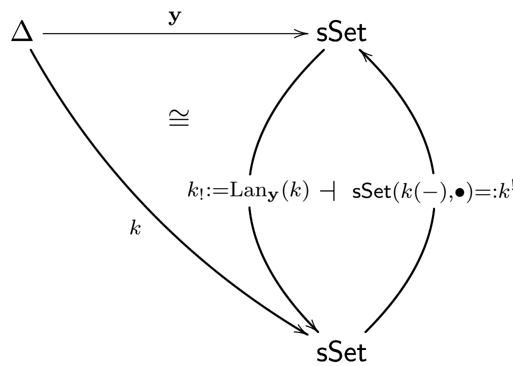
1.2. **Quillen equivalence between CSs and $(\mathsf{sSet})_{\mathrm{Joyal}}$.** Following [JT07], we present two Quillen equivalences between CSs and $(\mathsf{sSet})_{\mathrm{Joyal}}$. The flavour of the discussion will be quite categorically inclined

1.2.1. *Numerous adjunctions.* Consider the functor

$$k\colon \Delta \to \mathsf{sSet}, \qquad [n] \mapsto \Delta'[n], \tag{11}$$

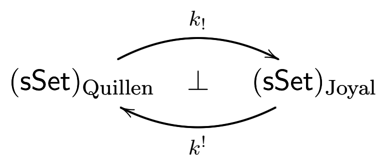
where $\Delta'[n]$ is (the nerve of) the free groupoid on the category $[n]$. The universal property of the Yoneda embedding $\mathbf{y}\colon \Delta \to \mathsf{sSet}$ gives

(12)



$k_!$ is obtained as a colimit-preserving extension of k to sSet.

Proposition 1.28 ([JT07], Prop 1.19)**.** *There is a Quillen pair*

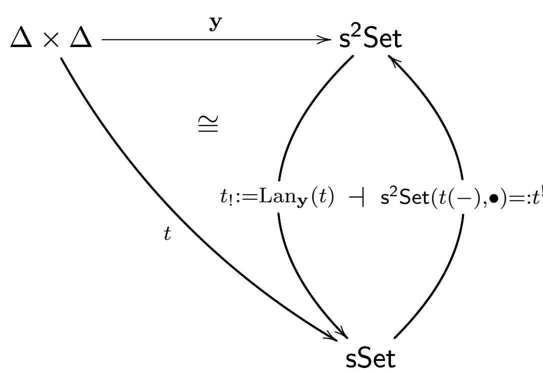


In a similar fashion, consider the functor

$$t\colon \Delta \times \Delta \to \mathsf{sSet}, \qquad ([m],[n]) \mapsto \Delta[m] \times \Delta'[n]. \tag{13}$$

The same $\mathbf{y}$oga as above gives

(14)



Since $\Delta[m]\Box\Delta[n] \cong (\Delta\times\Delta)(-,([m],[n]))$,

$$t_!(\Delta[m]\Box\Delta[n]) \cong \Delta[m] \times \Delta'[n].$$

We gather several useful interactions of $k_! \dashv k^!$ and $t_! \dashv t^!$ in the following

Lemma 1.29 ([JT07], Lemma 2.11)**.** *There are isomorphisms*

$$t_!(K\Box L) \cong K \times k_!(L), \quad K\backslash t^!(X) \cong k^!(X^K) \quad \text{and} \quad t^!(X)/L \cong X^{k_!(L)} \tag{15}$$

natural in $K, L \in \mathsf{sSet}$ *and* $X \in \mathsf{s^2Set}$*.*

Proof. Both functors

$$(K,L) \mapsto t_!(K\Box L) \quad \text{and} \quad (K,L) \mapsto K \times k_!(L)$$

are cocontinuous in each variable and coincide on the pairs $(\Delta[m],\Delta[n])$. The first natural isomorphism follows. The second isomorphism follows from the first one because, for a fixed $K \in \mathsf{sSet}$,

$X \mapsto k^!(X^K)$ is right adjoint to $L \mapsto K \times k_!(L)$ and $X \mapsto K\backslash t^!(X)$ is right adjoint to $L \mapsto K \times k_!(L)$, since right adjoints compose. In the same way, the third isomorphism holds because the two sides of it are right adjoint to the naturally isomorphic functors $K \mapsto t_!(K \square L)$ and $K \mapsto K \times k_!(L)$ (for a fixed $L \in \mathsf{sSet}$). $\square$

We can say more about the adjoint pair $t_! \dashv t^!$.

Theorem 1.30 ([JT07], Thm 2.12)**.** *There is a Quillen pair*

$$\mathsf{s^2Set} \underset{t^!}{\overset{t_!}{\rightleftarrows}} (\mathsf{sSet})_{\text{Joyal}} \quad (\perp)$$

where $\mathsf{s^2Set}$ *is given either the horizontal or the vertical model category structure.*

Proof. The proof of this result is a nice example of categorical homotopy theory in action. We treat the cases of the horizontal and the vertical model category structure on $\mathsf{s^2Set}$ separately.

(a) Let us first establish that $(t_!, t^!)$ is a Quillen pair when $\mathsf{s^2Set}$ has the horizontal model category structure. We need to check that $t_!$ preserves cofibrations and $t^!$ preserves fibrations.

- To show that $t_!$ sends monomorphism to monomorphism, by Proposition 2.11 it suffices to prove that $t_!(\delta_m \square' \delta_n)$ is a monomorphism, where $\delta_m \colon \partial\Delta[m] \hookrightarrow \Delta[m]$ is the boundary inclusion and $(\bullet)\square'(?)$ is the functor defined in Appendix, (31). But the map $t_!(\delta_m \square' \delta_n)$ is isomorphic to $\delta_m \times k_!(\delta_n)$ with $k_!(\delta_n)$ being a mono by Proposition 1.28.
- To show that $t^!(f)$ is a horizontal fibration for every Joyal fibration $f \colon X \to Y$, it is enough to show that $\langle t^!(f)/u \rangle$ is a Joyal fibration for every mono $u \colon K \to L$ in $\mathsf{s^2Set}$, by the definition of horizontal fibrations (see Theorem 2.15). But now, from (15), the map

$$\left\langle t^!(f)/u \right\rangle : t^!(X)/L \to \left(t^!(Y)/L\right) \times_{\left(t^!(Y)/K\right)} \left(t^!(X)/K\right)$$

is isomorphic to the map

$$\langle k_!(u), f \rangle : X^{k_!(L)} \to Y^{k_!(L)} \times_{Y^{k_!(K)}} X^{k_!(K)}.$$

Since $k_!(u)$ is a monomorphism, f is a Joyal fibration and $(\mathsf{sSet})_{\text{Joyal}}$ is a cartesian closed model category, we get that $\langle k_!(u), f \rangle$ is indeed a monomorphism.

This finishes the proof that $(t_!, t^!)$ is a Quillen pair for the horizontal model category structure.

(b) Let us now consider the case where $\mathsf{s^2Set}$ has the vertical model category structure. We only need to show that if $f \colon X \to Y$ is a Joyal fibration, then $t^!(f)$ is a vertical fibration. By Proposition 2.13, it is enough to show that

$$\left\langle u\backslash t^!(X) \right\rangle : L\backslash t^!(X) \to \left(L\backslash t^!(Y)\right) \times_{\left(K\backslash t^!(Y)\right)} \left(K\backslash t^!(X)\right)$$

is Kan fibration for every mono $u \colon K \to L$ in sSet. But, again by (15), that map is isomorphic to

$$k^! \langle u, f \rangle : k^!(X^L) \to k^!(Y^L) \times_{k^!(Y^K)} k^!(X^K)$$

which is a monomorphism, because $\langle u, f \rangle$ is a Joyal fibration and we can use Proposition 1.28.

$\square$

Theorem 1.31 ([JT07], Thm 3.3)**.** *The Quillen pair of Theorem 1.30 descends to a Quillen pair*

$$\mathsf{Ss} \underset{t^!}{\overset{t_!}{\rightleftarrows}} (\mathsf{sSet})_{\text{Joyal}} \qquad (\bot)$$

Proof. In light of Theorem 1.30, it suffices to show that $t^!$ takes a quasi-category to a Segal space. For, if this is the case, $t^!\colon (\mathsf{sSet})_{\text{Joyal}} \to \mathsf{Ss}$ takes fibrant objects to fibrant objects and we already know that it takes Joyal fibrations to vertical fibrations. Hence, $t^!$ takes fibrations between fibrant objects in $(\mathsf{sSet})_{\text{Joyal}}$ to fibrations between fibrant objects in Ss.
Take then a quasi-category X. We know that $t^!(X)$ is vertically fibrant by Theorem 1.30, so it is enough to show that, for every $m \in \mathbb{N}$, $\varphi^m \backslash t^!(X)$ is a trivial Kan fibration (see Remark 1.2). This map is isomorphic to $k^!(X^{\varphi^m})$ by (15). Now, φ^m is a trivial cofibration in $(\mathsf{sSet})_{\text{Joyal}}$, so X^{φ^m} is a trivial Kan fibration and therefore so is $k^!(X^{\varphi^m})$ by Proposition 1.28. This concludes our proof. □

Finally, there is an adjoint pair

$$\Delta \times \Delta \underset{i_1}{\overset{p_1}{\rightleftarrows}} \Delta \qquad (\bot)$$

where p_1 is the projection functor onto the first factor and i_1 sends $[n] \in \Delta$ to $([n], [0]) \in \Delta \times \Delta$. Precomposition gives then an adjoint pair

(16)
$$\mathsf{sSet} \underset{i_1^* = (-)\circ i_1}{\overset{p_1^* = (-)\circ p_1}{\rightleftarrows}} \mathsf{s^2Set} \qquad (\bot)$$

We get

(17)
$$p_1^*(-) \cong (-)\square\Delta[0] \cong c_v(-) \qquad \text{and} \qquad i_1^*(-) = (-)_{\bullet,0}$$

(see Appendix, (24) for the definition of the functor c_v).

1.2.2. *Two equivalences, in two directions.* We want to show that there are Quillen equivalences

(18)
$$(\mathsf{sSet})_{\text{Joyal}} \underset{i_1^*}{\overset{p_1^*}{\rightleftarrows}} \mathsf{CSs} \underset{t^!}{\overset{t_!}{\rightleftarrows}} (\mathsf{sSet})_{\text{Joyal}} \qquad (\bot,\ \bot)$$

We start by addressing the leftmost adjoint pair.

Lemma 1.32. *There is a Quillen pair*

$$(\mathsf{sSet})_{\text{Joyal}} \underset{i_1^*}{\overset{p_1^*}{\rightleftarrows}} \mathsf{CSs} \qquad (\bot)$$

Proof. Since $p_1^* = c_v$, it sends monomorphisms to monomorphisms, hence it preserves cofibrations. On the other hand, by Proposition 1.26 every fibration in CSs is a horizontal fibration and horizontal fibrations are row-wise Joyal fibrations. Since $i_1^*(-) = (-)_{\bullet,0}$, it then preserves fibrations. $\square$

In order to show $p_1^* \dashv i_1^*$ is a Quillen equivalence, we need to consider a special fibrant replacement in CSs of $p_1^*(X)$, for X a quasi-category.

Recall that there is a functor

$$J\colon \mathsf{QCat} \to \mathsf{Kan}$$

from the full subcategory of sSet spanned by quasi-categories to the full subcategory of sSet spanned by Kan complexes. It associates to each quasi-category its largest sub-Kan complex and takes Joyal equivalences (resp. Joyal fibrations) between quasi-categories to Kan-Quillen equivalences (resp. Kan fibrations) between Kan complexes (see [JT07], Prop 1.16). We then define a functor

$$\Gamma\colon \mathsf{QCat} \to \mathsf{s^2Set}, \qquad X \mapsto ([m] \mapsto J(X^{\Delta[m]})) \tag{19}$$

Remark 1.33. Γ is the quasi-categorical generalization of the classifying diagram functor (see Definition 1.8).

For X a quasi-category, we have

$$i_1^*\Gamma(X) = \Gamma(X)_{\bullet,0} \cong X,$$

By adjointness, we have a map $p_1^*(X) \to \Gamma(X)$. This is the sought fibrant approximation to $p_1^*(X)$.

Proposition 1.34 ([JT07], Prop 4.10)**.** *Given a quasi-category X, $\Gamma(X)$ is a complete Segal space and the natural map $p_1^*(X) \to \Gamma(X)$ is a weak equivalence in CSs.*

We are now ready to prove the

Theorem 1.35 ([JT07], Thm 4.11)**.** *There is a Quillen equivalence*

$$(\mathsf{sSet})_{\mathrm{Joyal}} \underset{i_1^*}{\overset{p_1^*}{\rightleftarrows}} \mathsf{CSs} \quad (\perp) \tag{20}$$

Proof. By Lemma 1.32, (20) is a Quillen pair. Since every object in $(\mathsf{sSet})_{\mathrm{Joyal}}$ is cofibrant, in order to conclude we need to show that

(*a*) for every complete Segal space X, the counit map $\epsilon\colon p_1^*i_1^*X \to X$ is a weak equivalence in CSs;

(*b*) for every quasi-category K, the map $K \to i_1^*Rp_1^*K$ (induced by the unit map) is a Joyal equivalence, where $p_1^*K \to Rp_1^*K$ is a fibrant replacement of p_1^*K in CSs.

Since CSs is a left Bousfield localization of the horizontal model structure on $\mathsf{s^2Set}$ (see Proposition 1.26), (*a*) follows if we can show that $\epsilon_{\bullet,n}\colon X_{\bullet,0} \to X_{\bullet,n}$ is a Joyal equivalence for all $n \in \mathbb{N}$ ($(p_1^*i_1^*X)_{\bullet,n} = (c_v(X_{\bullet,0}))_{\bullet,n} = X_{\bullet,0}$). But $\epsilon_{\bullet,n}$ is just the map $X_{\bullet,0} \to X_{\bullet,n}$ obtained from the unique map $[n] \to [0]$, so it is a Joyal equivalence because a (complete) Segal space is vertically fibrant, hence categorically constant by Proposition 2.18. Thus (*a*) holds. As for (*b*), a fibrant replacement of p_1^*K in CSs can be taken as $\Gamma(K)$ by Proposition 1.34. In this case, $Rp_1^*K \cong K$ and $K \to i_1^*Rp_1^*K$ is (isomorphic to) the identity map. This concludes the proof. $\square$

We are also now ready to show that the right adjunction in (18) is a Quillen equivalence.

Theorem 1.36 ([JT07], Thm 4.12)**.** *There is a Quillen equivalence*

$$\mathsf{CSs} \underset{t^!}{\overset{t_!}{\rightleftarrows}} (\mathsf{sSet})_{\text{Joyal}} \qquad \perp \tag{21}$$

Proof. It is enough to show that $(t_!, t^!)$ as in (21) is a Quillen pair. For, the composite $t_! p_1^* \colon \mathsf{sSet} \to \mathsf{sSet}$ is isomorphic to the identity functor since, for all $K \in \mathsf{sSet}$,

$$t_! p_1^*(K) \cong t_!(K \square \Delta[0]) \cong K \times k_!(\Delta[0]) \cong K,$$

thanks to (15) and the fact that $k_!(\Delta[0]) \cong \Delta_0$. By adjointness, $i_1^* t^!$ is also isomorphic to the identity functor. So, if $(t_!, t^!)$ is a Quillen pair, then it is a Quillen equivalence because Quillen equivalences satisfies the 2-out-of-3 property among Quillen pairs.
To show that $(t_!, t^!)$ in (21) is a Quillen pair, it is enough to show that $t^!$ carries quasi-categories to complete Segal spaces. If X is a quasi-category, then $t^!(X)$ is a Segal space, so we need to show that the map $u_0 \backslash t^!(X)$ is a trivial Kan fibration (see Proposition 1.21). But now, by (15), $u_0 \backslash t^!(X)$ is isomorphic to $k^!(X^{u_0})$ and this is a trivial Kan fibration thanks to Proposition 1.28 (since u_0 is a trivial cofibration). This completes the proof. ☐

2. Appendix: on bisimplicial sets

We collect here a lot of facts about bisimplicial sets and their homotopy theories.

2.1. Generalities.

Let $\mathsf{s}^2\mathsf{Set}$ be the category of bisimplicial sets. It can be described as:

- the category $\mathsf{sPrSh}(\Delta)$ of functors $\Delta^{\text{op}} \to \mathsf{sSet}$;
- the category $\mathsf{PrSh}(\Delta \times \Delta)$ of set-valued presheaves over the product category $\Delta \times \Delta$.

Given $X \in \mathsf{s}^2\mathsf{Set}$ and $m, n \in \mathbb{N}$, we write

$$X_m := X([m]) \qquad \text{and} \qquad X_{m,n} := (X_m)(n).$$

The elements of the set $X_{m,n}$ are the $(m,n)-$*bisimplices* of X. The simplicial sets

$$X_{m,\bullet} := X_m \qquad \text{and} \qquad X_{\bullet,n} \colon [m] \mapsto X_{m,n} \tag{22}$$

are called the *m-th column* and the *n-th row* of X respectively .

There are embeddings

$$c_h \colon \mathsf{sSet} \longrightarrow \mathsf{s}^2\mathsf{Set}, \qquad K \mapsto ([m] \mapsto K) \tag{23}$$

$$c_v \colon \mathsf{sSet} \longrightarrow \mathsf{s}^2\mathsf{Set}, \qquad K \mapsto (([m],[n]) \mapsto K_m) \tag{24}$$

They see a simplicial set as a bisimplicial set constant in horizontal degree and as a bisimplicial set constant in vertical degree respectively.

Definition 2.1. A bisimplicial set X is called *discrete* if there is a simplicial set K such that $c_v(K) \cong X$.

Convention 2.2. We will always consider a simplicial set K as a vertically constant bisimplicial set and just write K instead of $c_v(K)$.

2.2. **The vertical model category structure for $\mathsf{s}^2\mathsf{Set}$.** Since $(\mathsf{sSet})_{\mathrm{Quillen}}$ is a simplicial, proper and combinatorial model category, we can consider the injective model category structure on $(\mathsf{sSet})_{\mathrm{Quillen}}^{\Delta^{\mathrm{op}}}$. We thus get the following

Theorem 2.3. *The category $\mathsf{s}^2\mathsf{Set}$ of bisimplicial sets has a simplicial, proper and combinatorial model category structure for which a map $f\colon X \to Y$ of bisimplicial set is:*

- *a* weak equivalence *if and only if it is a vertical weak equivalence. This means that, for all $m \in \mathbb{N}$, the induced map $f_m\colon X_m \to Y_m$ of vertical simplicial sets is a weak equivalence in $(\mathsf{sSet})_{\mathrm{Quillen}}$;*
- *a* cofibration *if and only if it is a monomorphism;*
- *a* fibration *if and only if it has the right lifting property with respect to all maps that are weak equivalences and cofibrations.*

For $X, Y \in \mathsf{s}^2\mathsf{Set}$, the simplicial mapping space $\mathrm{Map}_{\mathsf{s}^2\mathsf{Set}}(X,Y)$ has n-simplices given by

$$\mathrm{Map}_{\mathsf{s}^2\mathsf{Set}}(X,Y)_n = \mathsf{s}^2\mathsf{Set}(X \times c_h(\Delta[n]),\ Y) \tag{25}$$

Definition 2.4. We call the model structure on bisimplicial sets of Theorem 2.3, the *vertical model structure* on $\mathsf{s}^2\mathsf{Set}$ and denote it by $(\mathsf{s}^2\mathsf{Set})_v$. We call the fibrations and the trivial fibrations of $(\mathsf{s}^2\mathsf{Set})_v$ the *vertical fibrations* and the *vertical trivial fibrations* respectively.

Remark 2.5. A vertical (trivial) fibration $f\colon X \to Y$ of bisimplicial set is a column-wise (trivial) Kan fibration, i.e. each map $f_{m,\bullet}$ is a (trivial) fibration in $(\mathsf{sSet})_{\mathrm{Quillen}}$.

$\mathsf{s}^2\mathsf{Set}$ is cartesian closed: given $X, Y \in \mathsf{s}^2\mathsf{Set}$, the internal hom Y^X can be described as

$$(Y^X)_m = \mathrm{Map}_{\mathsf{s}^2\mathsf{Set}}(X \times \Delta[m],\ Y), \tag{26}$$

for all $m \in \mathbb{N}$ (recall Convention 2.2). Notice that $(Y^X)_0 \cong \mathrm{Map}_{\mathsf{s}^2\mathsf{Set}}(X,Y)$.

Proposition 2.6. *The category of bisimplicial sets with the vertical model structure of Theorem 2.3 is a* cartesian closed model category. *This means that the terminal object in $(\mathsf{s}^2\mathsf{Set})_v$ is cofibrant and, for every pair of cofibrations $i\colon A \rightarrowtail B$ and $j\colon C \rightarrowtail D$ and for every fibration $p\colon X \twoheadrightarrow Y$ in $(\mathsf{s}^2\mathsf{Set})_v$, the following equivalent properties hold:*

(1) the pushout-product map

$$(A \times D) \coprod_{A \times C} (B \times C) \to B \times D$$

is a cofibration. It is also a weak equivalence if either of i or j is;

(2) the pullback-exponential map

$$Y^B \to Y^A \times_{X^A} X^B$$

is a fibration. It is also a weak equivalence if either i or p is.

2.3. **On vertical (trivial) fibrations.** The simplex category Δ is an *elegant Reedy category* (see [BR11], Def 3.5) and the model structure $(\mathsf{s}^2\mathsf{Set})_v$ is also the Reedy model structure on $(\mathsf{sSet})_{\mathrm{Quillen}}^{\Delta^{\mathrm{op}}}$. Practically speaking, this means we can rely on a nice(r) description of the vertical (trivial) fibrations and get more information about the vertically fibrant objects.

Definition 2.7. Let K and L be simplicial sets. We define their *box product* (or *external product*) to be the bisimplicial set $K \square L$ given by

$$(K \square L)_{m,n} := K_m \times L_n,$$

with the obvious action on maps.

The assignment $(K, L) \mapsto K\square L$ extends to a functor

$$(\bullet)\square(?) \colon \mathsf{sSet} \times \mathsf{sSet} \to \mathsf{s^2Set} \tag{27}$$

Note that, for $k, l \in \mathbb{N}$, $\Delta[k]\square\Delta[l] \cong (\Delta \times \Delta)(-,\ ([k], [l]))$.

The box product bifunctor has right adjoints in both variables. Namely, let K be a simplicial set; then:

- the functor $K\square(\bullet) \colon \mathsf{sSet} \to \mathsf{s^2Set}$ has a right adjoint
$$K\backslash(\bullet) \colon \mathsf{s^2Set} \to \mathsf{sSet}, \quad X \mapsto \mathsf{s^2Set}(K\square\Delta[-], X). \tag{28}$$
Note that, for $X \in \mathsf{s^2Set}$ and $m \in \mathbb{N}$, $\Delta[m]\backslash X \cong X_{m,\bullet}$, the $m-$th column of X.
- the functor $(\bullet)\square K \colon \mathsf{sSet} \to \mathsf{s^2Set}$ has a right adjoint
$$(\bullet)/K \colon \mathsf{s^2Set} \to \mathsf{sSet}, \quad X \mapsto \mathsf{s^2Set}(\Delta[-]\square K, X). \tag{29}$$
Note that, for $X \in \mathsf{s^2Set}$ and $n \in \mathbb{N}$, $X/\Delta[n] \cong X_{\bullet,n}$, the $n-$th row of X.

Thus, for $K, L \in \mathsf{sSet}$ and $X \in \mathsf{s^2Set}$, we have natural isomorphisms

$$\mathsf{s^2Set}(K\square L, X) \cong \mathsf{s^2Set}(L, K\backslash X) \cong \mathsf{s^2Set}(A, X/B). \tag{30}$$

Remark 2.8. We actually get *bifunctors*

$$(\bullet)\backslash(?) \colon \mathsf{sSet}^{\mathrm{op}} \times \mathsf{s^2Set} \to \mathsf{sSet} \quad \text{and} \quad (?)/(\bullet) \colon \mathsf{s^2Set} \times \mathsf{sSet}^{\mathrm{op}} \to \mathsf{sSet}.$$

We can now run the *Leibniz construction* machinery (see [RV14]) to obtain a bifunctor

$$\begin{gathered}(\bullet)\square'(?) \colon \mathsf{sSet}^{\bullet\to\bullet} \times \mathsf{sSet}^{\bullet\to\bullet} \to \mathsf{s^2Set}^{\bullet\to\bullet} \\ (u \colon K \to L,\ v \colon S \to T) \mapsto (K\square T \amalg_{K\square S} L\square S \to L\square T)\end{gathered} \tag{31}$$

where $\mathcal{C}^{\bullet\to\bullet}$ denotes the arrow category of a category $\mathcal{C}$. For a fixed map $u \colon K \to L$ of simplicial sets, we can similarly define functors

$$\langle u\backslash(\bullet)\rangle \colon \mathsf{s^2Set}^{\bullet\to\bullet} \to \mathsf{sSet}^{\bullet\to\bullet}, \quad (f \colon X \to Y) \mapsto \big(L\backslash X \to L\backslash Y \times_{K\backslash Y} K\backslash X\big) \tag{32}$$

and

$$\langle(\bullet)/u\rangle \colon \mathsf{s^2Set}^{\bullet\to\bullet} \to \mathsf{sSet}^{\bullet\to\bullet}, \quad (f \colon X \to Y) \mapsto \big(X/L \to Y/L \times_{Y/K} X/K\big) \tag{33}$$

As above, we get adjoint pairs

$$u\square'(\bullet) \dashv \langle u\backslash(\bullet)\rangle \quad \text{and} \quad (\bullet)\square' u \dashv \langle(\bullet)/u\rangle\,.$$

Remark 2.9. Let $X \in \mathsf{s^2Set}$, $K \in \mathsf{sSet}$, $g \colon Y \to Z$ a map in $\mathsf{s^2Set}$ and $v \colon S \to T$ a map in sSet. It follows that

$$X/v \cong \langle(X \to 1)/v\rangle \quad \text{and} \quad K\backslash g \cong \langle(\emptyset \to K)\backslash g\rangle$$

where $\emptyset$ and 1 denote the initial simplicial set and the terminal bisimplicial set respectively.

A somewhat standard adjointness argument proves the following

Lemma 2.10. *For maps $u, v \in \mathsf{sSet}$ and $f \in \mathsf{s^2Set}$,*

$$(u\square' v) \boxslash f \iff u \boxslash \langle f/v\rangle \iff v \boxslash \langle u\backslash f\rangle$$

Let us denote by δ_n the boundary inclusion $\partial\Delta[n] \subseteq \Delta[n]$ and by h_n^k the horn inclusion $\Lambda^k[n] \subseteq \Delta[n]$.

Proposition 2.11. *The saturations of the sets of maps*

$$\delta_m\square' h_n^k, \quad m \geq 0,\ k \geq n \leq 0 \tag{34}$$

and

$$\delta_m\square'\delta_n, \quad m, n \geq 0 \tag{35}$$

are given by the class of trivial cofibrations and of cofibrations in $(\mathsf{s^2Set})_v$ respectively.

The above Proposition together with Lemma 2.10 imply the following characterizations of vertical (trivial) fibrations.

Proposition 2.12 ([JT07], Prop. 2.3)**.** *The following are equivalent, for a map $f\colon X \to Y$ of bisimplicial set.*

(i) *f is a vertical trivial fibration;*
(ii) *f has the right lifting property with respect to the maps in (35);*
(iii) *$\langle \delta_m \backslash f\rangle$ is a trivial Kan fibration for every $m \in \mathbb{N}$;*
(iv) *$\langle u \backslash f\rangle$ is a trivial Kan fibration for every monomorphism u in sSet;*
(v) *$\langle f/\delta_n\rangle$ is a trivial Kan fibration for every $n \in \mathbb{N}$;*
(vi) *$\langle f/v\rangle$ is a trivial Kan fibration for every monomorphism $v \in \mathsf{sSet}$.*

Proposition 2.13 ([JT07], Prop. 2.5)**.** *The following are equivalent, for a map $f\colon X \to Y$ of bisimplicial set.*

(i) *f is a vertical fibration;*
(ii) *f has the right lifting property with respect to the maps in (34);*
(iii) *$\langle \delta_m \backslash f\rangle$ is a Kan fibration for every $m \in \mathbb{N}$;*
(iv) *$\langle u \backslash f\rangle$ is a Kan fibration for every monomorphism u in sSet;*
(v) *$\left\langle f/h_n^k\right\rangle$ is a trivial Kan fibration for every $n \in \mathbb{N}$;*
(vi) *$\langle f/v\rangle$ is a trivial Kan fibration for every trivial cofibration $v \in (\mathsf{sSet})_{\mathrm{Quillen}}$.*

Lemma 2.14. *If X is a vertically fibrant bisimplicial set, then each X_m is a Kan complex and the map*

$$(d_0, d_1)\colon X_1 \to X_0 \times X_0$$

is a Kan fibration. In particular, each of the maps $d_0, d_1\colon X_1 \to X_0$ are Kan fibrations.

2.4. **The horizontal model category structure for $\mathsf{s}^2\mathsf{Set}$.** Maps of bisimplicial sets which give row-wise weak equivalences in the *Joyal model structure* on sSet (see [Joy], Chapter 6) are the weak equivalences for a model structure on $\mathsf{s}^2\mathsf{Set}$.

Theorem 2.15 ([JT07], Prop 2.10)**.** *$\mathsf{s}^2\mathsf{Set}$ admits a model structure $(\mathsf{s}^2\mathsf{Set})_h$ for which a map $f\colon X \to Y$ of bisimplicial sets is*

- *a weak equivalence if and only if it is a horizontal equivalence. This means that, for all $n \in \mathbb{N}$, the map $f_{\bullet,n}\colon X_{\bullet,n} \to Y_{\bullet,n}$ is a weak equivalence in $(\mathsf{sSet})_{\mathrm{Joyal}}$, i.e. a weak categorical equivalence;*
- *a cofibration if and only if it is a monomorphism;*
- *a fibration if and only if it is a horizontal fibration, that is if $\langle f/\delta_n\rangle$ is a Joyal fibration for every $n \in \mathbb{N}$ (here $\delta_n\colon \partial\Delta[n] \hookrightarrow \Delta[n]$ is the boundary inclusion).*

The model structure $(\mathsf{s}^2\mathsf{Set})_h$ is left proper and cartesian closed.

Definition 2.16. We call the model structure of Theorem 2.15 the *horizontal model structure* on $\mathsf{s}^2\mathsf{Set}$ and call its fibrant objects *horizontally fibrant* bisimplicial sets.

There is some interplay between the vertical and the horizontal model structure on $\mathsf{s}^2\mathsf{Set}$.

Definition 2.17. A bisimplicial set X is *categorically constant* if the canonical map

$$X_{\bullet,n} \to X_{\bullet,0},$$

induced by $[n] \to [0]$, is a Joyal equivalence for all $n \in \mathbb{N}$.

Proposition 2.18 ([JT07], Prop 2.8 & Prop 2.9)**.**

(i) *A vertically fibrant simplicial set is categorically constant.*

(ii) *A map $f\colon X \to Y$ of vertically fibrant simplicial set is a horizontal equivalence if and only if it induces a Joyal equivalence between the first rows.*

Proof. Recall that, for every $n \in \mathbb{N}$ and every $X \in \mathsf{s^2Set}$, $X/\Delta[n]$ is isomorphic to the n-th row of X. If X is vertically fibrant, by Proposition 2.12 and Remark 2.9, the map

$$X/(\Delta[0] \hookrightarrow \Delta[n])\colon X/\Delta[n] \to X/\Delta[0]$$

is a trivial Kan fibration, hence also a trivial fibration in the Joyal model structure (see Example 2.20 below). Since

$$(\Delta[n] \to \Delta[0]) \circ (\Delta[0] \hookrightarrow \Delta[n]) = \mathrm{id}_{\Delta[0]},$$

the same is true after applying $X/(\bullet)$, with $X/\,\mathrm{id}_{\Delta[0]} = \mathrm{id}_{X/\Delta[0]}$. By 2-out-of-3, $X/(\Delta[n] \hookrightarrow \Delta[0])$ is a Joyal equivalence. This shows (i). The second claim is obtained by looking at the commutative diagrams, for every $n \in \mathbb{N}$,

$$\begin{array}{ccc} X_{\bullet,0} & \xrightarrow{f_{\bullet,0}} & Y_{\bullet,0} \\ \downarrow & & \downarrow \\ X_{\bullet,n} & \xrightarrow[f_{\bullet,n}]{} & Y_{\bullet,n} \end{array}$$

where the vertical maps are canonically induced by $\Delta[n] \to \Delta[0]$. □

2.5. **Left Bousfield Localizations.**

Definition 2.19. Let $\mathcal{M}$, $\mathcal{M}'$ be model categories with the same underlying category. Let $\mathcal{W}$, $\mathcal{F}$ and $\mathcal{C}$ be the classes of weak equivalences, fibrations and cofibrations in $\mathcal{M}$ respectively. Similary, denote by $\mathcal{W}'$, $\mathcal{F}'$ and $\mathcal{C}'$ the classes of weak equivalences, fibrations and cofibrations in $\mathcal{M}'$ respectively. We say that the model category $\mathcal{M}'$ is *a left Bousfield localization* of the model category $\mathcal{M}$ if $\mathcal{W} \subseteq \mathcal{W}'$ and $\mathcal{C} = \mathcal{C}'$.

Example 2.20. The Kan-Quillen model structure on simplicial sets is a left Bousfield localization of the Joyal model structure on simplicial sets (see [JT07], Prop 1.15).

Keeping the same notations as in Definition 2.19, we need to have $\mathcal{F}' \subseteq \mathcal{F}$, whereas the trivial fibrations must be the same in $\mathcal{M}$ and in $\mathcal{M}'$. The difference between the fibrations and the weak equivalences in $\mathcal{M}$ and in $\mathcal{M}'$ is only witnessed by the non-fibrant objects of the latter, as explained by the following

Proposition 2.21 ([JT07], Prop 7.21)**.** *Let $\mathcal{M}'$ be a left Bousfield localization of a model category $\mathcal{M}$. Then a map between $\mathcal{M}'$-fibrant objects is a fibration (resp. a weak equivalence) in $\mathcal{M}'$ if and only if it is a fibration (resp. a weak equivalence) in $\mathcal{M}$.*

There is a general machinery to produce left Bousfield localizations out of sets of maps in a model category $\mathcal{M}$. We describe it here in the specific case when $\mathcal{M}$ is $(\mathsf{s^2Set})_v$. The general theory can be found in [Hir03], Chapter 3.

Given a *set* S of maps in $\mathsf{s^2Set}$, we say that a bisimplicial set Z is *(vertically) S-local* if it is vertically fibrant and, for every map $s\colon U \to V$ in S, the induced map on function complexes

$$\mathrm{Map}_{\mathsf{s^2Set}}(s, Z)\colon \mathrm{Map}_{\mathsf{s^2Set}}(V, Z) \to \mathrm{Map}_{\mathsf{s^2Set}}(U, Z) \tag{36}$$

is a Kan-Quillen equivalence of simplicial sets. Furthermore, we say that a map $f\colon X \to Y$ of bisimplicial sets is a *(vertical) S-local equivalence* if, for all $S-$local bisimplicial set Z, the induced map

$$\mathrm{Map}_{\mathsf{s^2Set}}(f, Z)\colon \mathrm{Map}_{\mathsf{s^2Set}}(Y, Z) \to \mathrm{Map}_{\mathsf{s^2Set}}(X, Z) \tag{37}$$

is a Kan-Quillen equivalence of simplicial sets.

Remark 2.22. All maps in S and all vertical equivalences are S-local equivalences.

Theorem 2.23. *Let S be a set of maps of bisimplicial sets. Then there is a left proper, simplicial and combinatorial model category, denoted by $\mathcal{L}_S((\mathsf{s}^2\mathsf{Set})_v)$, having $\mathsf{s}^2\mathsf{Set}$ as the underlying category. A map $f\colon X \to Y$ of bisimplicial sets is:*

- *a weak equivalence in $\mathcal{L}_S((\mathsf{s}^2\mathsf{Set})_v)$ if and only if it is an $S-$local equivalence;*
- *a cofibration in $\mathcal{L}_S((\mathsf{s}^2\mathsf{Set})_v)$ if and only if it is a monomorphism;*
- *a fibration in $\mathcal{L}_S((\mathsf{s}^2\mathsf{Set})_v)$ if and only if it has the right lifting property with respect to all $S-$local equivalences which are also cofibrations.*

The above result is a special case of [Hir03], Thm 4.1.1.

We call the model category $\mathcal{L}_S((\mathsf{s}^2\mathsf{Set})_v)$ *the left Bousfield localization* of $(\mathsf{s}^2\mathsf{Set})_v$ *at* S. By Remark 2.22, $\mathcal{L}_S((\mathsf{s}^2\mathsf{Set})_v)$ is indeed a left Bousfield localization of $(\mathsf{s}^2\mathsf{Set})_v$ (in the sense of Definition 2.19).

Proposition 2.24 ([Hir03], Prop. 3.4.1)**.** *Let S be a set of maps of bisimplicial sets and let $\mathcal{L}_S(\mathsf{s}^2\mathsf{Set}_v)$ be the left Bousfield localization at S. Then:*

- *a bisimplicial set X is fibrant in $\mathcal{L}_S((\mathsf{s}^2\mathsf{Set})_v)$ if and only if it is an $S-$local object;*
- *a map $f\colon X \to Y$ of $S-$local objects is a weak equivalence (resp. a fibration) in $\mathcal{L}_S(\mathsf{s}^2\mathsf{Set}_v)$ if and only if it is a weak equivalence (resp. a fibration) in $(\mathsf{s}^2\mathsf{Set})_v$;*
- *for $X, Y \in \mathsf{s}^2\mathsf{Set}$,*

$$\mathrm{Map}_{\mathcal{L}_S((\mathsf{s}^2\mathsf{Set})_v)}(X, Y) = \mathrm{Map}_{\mathsf{s}^2\mathsf{Set}}(X, Y).$$

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