MODELS FOR $(\infty, 1)$-CATEGORIES: PART 2

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1. Simplicial Categories: the natural model

For every $n \in \mathbb{N}$, an ∞-category should come equipped with (at least):

- a notion of n-morphism;
- a composition rule for n-morphisms;
- identity n-morphisms.

Simplicial categories (meaning simplicially enriched categories) are a natural way to accomplish this. Indeed, if $\mathbb{C}$ is a simplicial category, one has, for every $n \in \mathbb{N}$, $(n+1)$-morphisms given by the n-simplices of the mapping spaces $\mathrm{Map}_{\mathbb{C}}(x, y)$ (for $x, y \in \mathrm{Ob}(\mathbb{C})$) and $\mathbb{C}$ carries a composition law. A simplicial category $\mathbb{C}$ is also a natural example of a homotopy theory (in the sense specified in the Introduction) because it automatically comes with mapping spaces, whose set of path components can be used to define the homotopy category of $\mathbb{C}$.

If $\mathbb{C}$ is *locally Kan*, i.e. each mapping space $\mathrm{Map}_{\mathbb{C}}(x, y)$ is a Kan complex, then:

- all $n-$morphisms of $\mathbb{C}$ are (homotopically) invertible, for $n \geq 2$;
- $\mathbb{C}$ is enriched over ∞-$\mathsf{Gpd} = (\infty, 0)$-$\mathsf{Cat}$.

Thus, locally Kan simplicial categories ought to be models for $(\infty, 1)$-categories. The work of [Ber07] and [Lur09] shows that there is a model category structure on the category SCat of simplicial categories for which the fibrant objects are exactly the locally Kan simplicial categories. Furthermore, SCat with this model structure is Quillen equivalent to $(\mathsf{sSet})_{\mathrm{Joyal}}$.

1.1. The model category structure on SCat.

Definition 1.1. A *simplicial category* is a $(\mathsf{sSet}, \times, \Delta[0])$-enriched category.

A small simplicial category $\mathbb{C}$ comes with a *set of objects* $\mathrm{Ob}(\mathbb{C})$ and, for $x, y \in \mathrm{Ob}(\mathbb{C})$, with a *mapping space* $\mathrm{Map}_{\mathbb{C}}(x, y)$. We also have a composition $\circ\colon \mathrm{Map}_{\mathbb{C}}(z, y) \times \mathrm{Map}_{\mathbb{C}}(x, y) \to \mathrm{Map}_{\mathbb{C}}(x, z)$ and an identity map $\mathrm{id}_x \colon x \to x$, for every $x, y, z \in \mathrm{Ob}(\mathbb{C})$.

Small simplicial categories and simplicial functors between them form a category, denoted by SCat.

Proposition 1.2 ([Ber07], Prop 3.3)**.** SCat *is a complete and cocomplete category. Furthermore, the object functor* $\mathrm{Ob}\colon \mathsf{SCat} \to \mathsf{Set}$ *commutes with (co)limits.*

Recall the canonical model structure on Cat.

Theorem 1.3. *There is a model category structure on the category* Cat *of small categories, called the canonical model category structure on* Cat*, for which a functor* $f\colon \mathcal{C} \to \mathcal{D}$ *is*

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(1) a weak equivalence if and only if it is an equivalence of categories;
(2) a cofibration if and only if it is injective on objects;
(3) a fibration if and only if it is an isofibration, that is, for every object $c \in \mathcal{C}$ and every isomorphism $t\colon f(c) \to d$ in $\mathcal{D}$, there is an isomorphism $i\colon c \to c'$ in $\mathcal{C}$ such that $f(i) = t$.

This result is folklore.[1] Note that in this model category structure every small category is both fibrant and cofibrant. There are adjunctions

$$
\begin{tikzcd}
\mathsf{Cat} \arrow[rr, "D" description] \arrow[rr, phantom, bend left=25, "\perp"] \arrow[rr, phantom, bend right=25, "\perp"] & & \mathsf{SCat} \arrow[ll, bend right=50, "(-)_0"'] \arrow[ll, bend left=50, "\pi_0"]
\end{tikzcd} \tag{1}
$$

Here:

- D sees every small category $\mathcal{C}$ as a discrete simplicial category. This functor is fully faithful, so we will just write $\mathcal{C}$ in place of $D\mathcal{C}$;
- for $\mathbb{C} \in \mathsf{SCat}$, $\mathbb{C}_0$ is the category having the same objects and the same maps of $\mathbb{C}$;
- π_0 assigns to $\mathbb{C} \in \mathsf{SCat}$ its category of *path-components*.

Definition 1.4. The *category of (path-)components* of $\mathbb{C} \in \mathsf{SCat}$ is the category $\pi_0(\mathbb{C})$ with

- $\mathrm{Ob}(\pi_0(\mathbb{C})) := \mathrm{Ob}(\mathbb{C})$;
- for every $x, y \in \mathrm{Ob}(\pi_0(\mathbb{C}))$, $\pi_0(\mathbb{C})(x, y) := \pi_0(\mathrm{Map}_{\mathbb{C}}(x, y))$;
- composition inherited from the composition law of $\mathbb{C}$, since $\pi_0\colon \mathsf{sSet} \to \mathsf{Set}$ preserves binary products.

Definition 1.5. A map $t\colon x \to y$ in a simplicial category $\mathbb{C}$ is a *homotopy equivalence* if it becomes an isomorphism in $\pi_0(\mathbb{C})$.

Definition 1.6. A simplicial functor $f\colon \mathbb{C} \to \mathbb{D}$ between simplicial categories is a *Dwyer-Kan equivalence* (or a *DK equivalence* for short) if the following two conditions are satisfied:

(DK1) for every $x, y \in \mathrm{Ob}(\mathbb{C})$, $f_{x,y}\colon \mathrm{Map}_{\mathbb{C}}(x, y) \to \mathrm{Map}_{\mathbb{D}}(fx, fy)$ is a Kan-Quillen equivalence;
(DK2) the induced functor on categories of components $\pi_0(f)\colon \pi_0(\mathbb{C}) \to \pi_0(\mathbb{D})$ is essentially surjective.

Remark 1.7. (DK1) says that f is homotopically fully faithful. In presence of (DK1), condition (DK2) implies that $\pi_0(f)$ is an equivalence of categories.

Remark 1.8. If $f\colon \mathcal{C} \to \mathcal{D}$ is a functor between ordinary categories, then (DK1) says that f is fully faithful, whereas (DK2) says that $\pi_0(f) \cong f$ is essentially surjective.

Theorem 1.9 ([Ber07], Thm 1.1)**.** *There is a right proper and cofibrantly generated model category structure on* SCat *for which a simplicial functor $f\colon \mathbb{C} \to \mathbb{D}$ is:*

- *a weak equivalence if and only if it is a Dwyer-Kan equivalence;*
- *a fibration if and only if it satisfies the following two properties:*
  (F1) *for every $x, y \in \mathbb{C}$, $f_{x,y}\colon \mathrm{Map}_{\mathbb{C}}(x, y) \to \mathrm{Map}_{\mathbb{D}}(fx, fy)$ is a Kan fibration;*
  (F2) *$\pi_0(f)\colon \pi_0(\mathbb{C}) \to \pi_0(\mathbb{D})$ is an isofibration;*

[1] Still, the reader can look at https://ncatlab.org/nlab/show/canonical+model+structure+on+Cat for a detailed account.

- *a cofibration if and only if it has the left lifting property with respect to all simplicial functors wich are both weak equivalences and fibrations.*

We call the model category structure of Theorem 1.9 above the *Bergner model category structure* on SCat.

Remark 1.10. Since every functor $f\colon \mathcal{C} \to \mathcal{D}$ between ordinary categories satisfies (F1), (F2) ensures that f is a fibration in Cat if and only if it is a fibration in SCat. Thus, the model category structure induced on $\mathsf{Cat} \subseteq \mathsf{SCat}$ by the adjoint pair $(-)_0 \dashv D$ is the canonical one (see also Remark 1.7).

Remark 1.11. A small simplicial category $\mathbb{C}$ is fibrant in the model category of Theorem 1.9 is fibrant if and only if it is *locally Kan* in the sense that, for all $x, y \in \mathrm{Ob}(\mathbb{C})$, $\mathrm{Map}_{\mathbb{C}}(x,y)$ is a Kan complex.

The proof of Theorem 1.9 is slightly technical, so we will omit it. Let us instead take a look at what the generating cofibrations for SCat should be. There is a functor

$$U\colon \mathsf{sSet} \to \mathsf{SCat} \tag{2}$$

where, for $K \in \mathsf{sSet}$, UK is the simplicial category with $\mathrm{Ob}(UK) := \{0,1\}$, $\mathrm{Map}_{UK}(0,1) := K$, $\mathrm{Map}_{UK}(0,0) := \mathrm{Map}_{UK}(1,1) := \Delta[0]$ and $\mathrm{Map}_{UK}(1,0) := \emptyset$.

Remark 1.12. U has the property that, for any simplicial category $\mathbb{C}$ and any simplicial set K, the data of a simplicial functor $UK \to \mathbb{C}$ amount exactly to the choice of objects x, y in $\mathbb{C}$ and of a map $K \to \mathrm{Map}_{\mathbb{C}}(x,y)$ in sSet.

Suppose now that $f\colon \mathbb{C} \to \mathbb{D}$ is a trivial fibration in SCat. Then, for every $x, y \in \mathbb{C}$ and any $n \in \mathbb{N}$, every solid diagram below in sSet admits a dotted lift

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbb{C}}(x,y) \arrow[d, "f_{x,y}"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & \mathrm{Map}_{\mathbb{D}}(fx,fy) \end{tikzcd}$$

because $f_{x,y}$ is a trivial Kan fibration. Equivalently, every solid diagram below in SCat admits a dotted lift

$$\begin{tikzcd} U\partial\Delta[n] \arrow[r] \arrow[d] & \mathbb{C} \arrow[d, "f"] \\ U\Delta[n] \arrow[r] \arrow[ur, dashed] & \mathbb{D} \end{tikzcd}$$

so that f must have the right lifting property with respect to all maps $U\partial\Delta[n] \to U\Delta[n]$, for $n \in \mathbb{N}$. Since f is a DK equivalence for all objects $d \in \mathbb{D}$ there is a homotopy equivalence $s\colon fx \to d$ in $\mathbb{D}$, for some $x \in \mathbb{C}$. By (F2), there is a homotopy equivalence $t\colon x \to x'$ in $\mathbb{C}$ such that $f(t) = s$. In particular, $d = fx'$, so f must be onto on objects. Equivalently, f must have the right lifting property with respect to the unique map $\emptyset \to \mathbb{1}$, where $\emptyset$ and $\mathbb{1}$ denote the initial and the terminal simplicial category respectively. We then get the following

Lemma 1.13 ([Ber07], Prop 3.2)**.** *A map $f\colon \mathbb{C} \to \mathbb{D}$ is a trivial fibration in SCat if and only if it has the right lifting property with respect to $\emptyset \to \mathbb{1}$ and to all the maps $U\partial\Delta[n] \to U\Delta[n]$, for $n \in \mathbb{N}$.*

One can give a similar characterization for a set of generating acyclic cofibrations in SCat, though this is slightly more convoluted. See [Ber07], Section 2.

1.2. SCat **as a presentation of** $(\infty,1)$**-categories.** Recall the construction of the *homotopy coherent nerve* functor that we saw in Alex's talk. There is a cosimplicial object

$$\mathfrak{C}\colon \Delta \to \mathsf{SCat}$$

such that, for $n \in \mathbb{N}$, $\mathfrak{C}_n := \mathfrak{C}([n])$ is the simplicial category with:

- $\mathrm{Ob}(\mathfrak{C}_n) = \{0, \cdots, n\}$;
- for every $0 \le i, j \le n$, if $i > j$, then $\mathrm{Map}_{\mathfrak{C}_n}(i,j) = \emptyset$, otherwise $\mathrm{Map}_{\mathfrak{C}_n}(i,j)$ is the nerve of the poset

$$P_{ij} := \{I \subseteq \{0, \dots, n\} : \ i, j \in I \text{ and } \forall l \in I \ (i \le l \le j)\};$$

- when $i \le k \le j$ in $\{0, \dots, n\}$, the composition law on $\mathfrak{C}_n$ is given by

$$P_{i,k} \times P_{k,j} \to P_{i,j}, \qquad (I_1, I_2) \mapsto I_1 \cup I_2.$$

We then get an induced adjunction

$$\begin{tikzcd}
\Delta \arrow[r, "\mathbf{y}"] \arrow[dr, bend right, "\mathfrak{C}"'] \arrow[dr, phantom, bend left=20, "\cong"] & \mathsf{sSet} \arrow[d, bend right=60, "\mathfrak{C}=\mathrm{Lan}_{\mathbf{y}}(\mathfrak{C})"'] \arrow[d, phantom, "\dashv"] \\
& \mathsf{SCat} \arrow[u, bend right=60, "\mathsf{SCat}(\mathfrak{C}_{(-)},\bullet)=:N_{\mathrm{coh}}"']
\end{tikzcd} \tag{3}$$

The right adjoint $N_{\mathrm{coh}} = \mathsf{SCat}(\mathfrak{C}_{(-)}, \bullet)$ is the *homotopy coherent nerve* functor, whereas the left adjoint $\mathfrak{C}\colon \mathsf{sSet} \to \mathsf{SCat}$ is the *categorification* (or *rigidification*) functor. Following [Lur09] and [DS11], the goal is now to show $\mathfrak{C} \dashv N_{\mathrm{coh}}$ is a Quillen equivalence for the Joyal model category structure on sSet.

For every $\mathbb{C} \in \mathsf{SCat}$, let us denote by

$$\epsilon = \epsilon_{\mathbb{C}}\colon \mathfrak{C}N_{\mathrm{coh}}\mathbb{C} \to \mathbb{C}$$

the counit map for the adjunction (3). There is a crucial property of this counit map that we saw in Aji's talk:

Theorem 1.14. *For every fibrant* $\mathbb{C} \in \mathsf{SCat}$*, the counit map*

$$\epsilon_{\mathbb{C}}\colon \mathfrak{C}N_{\mathrm{coh}}\mathbb{C} \to \mathbb{C}$$

is a Dwyer-Kan equivalence of simplicial categories.

Proof. [Lur09], Thm 2.2.0.1 implies that $\epsilon_{\mathbb{C}}$ satisfies (DK1). (DK2) then follows because ϵ is the identity on objects:

$$\begin{aligned}
\mathrm{Ob}(\mathfrak{C}N_{\mathrm{coh}}\mathbb{C}) &\cong \mathrm{Ob}\left(\mathrm{colim}_{\Delta[n]\to N_{\mathrm{coh}}\mathbb{C}}\, \mathfrak{C}(\Delta[n])\right) \\
&\cong \mathrm{colim}_{\Delta[n]\to N_{\mathrm{coh}}\mathbb{C}}\, \mathrm{Ob}(\mathfrak{C}(\Delta[n])) \\
&\cong \mathrm{colim}_{\Delta[n]\to N_{\mathrm{coh}}\mathbb{C}}\, [n] = \mathrm{colim}_{\Delta[n]\to N_{\mathrm{coh}}\mathbb{C}}(\Delta[n])_0 \\
&\cong \left(\mathrm{colim}_{\Delta[n]\to N_{\mathrm{coh}}\mathbb{C}}\, \Delta[n]\right)_0 \\
&\cong (N_{\mathrm{coh}}\mathbb{C})_0 \\
&\cong \mathrm{Ob}(\mathbb{C}).
\end{aligned}$$

□

Alex showed that, if $\mathbb{C}$ is a locally Kan simplicial category, then $N_{\mathrm{coh}}(\mathbb{C})$ is a quasi-category. The same proof he gave also shows the following

Proposition 1.15 ([Lur09], Prop 1.1.5.10 & Rmk 1.1.5.11)**.** *The homotopy coherent nerve*

$$N_{\mathrm{coh}}\colon \mathsf{SCat} \to \mathsf{sSet}$$

sends every fibration $f\colon \mathbb{C} \to \mathbb{D}$ in SCat to a Joyal fibration.

Proposition 1.16 ([DS11], Thm 8.1)**.** *The following are equivalent for a map $g\colon K \to L$ of simplicial sets:*

(1) g is an equivalence in $(\mathsf{sSet})_{Joyal}$;
(2) $\mathfrak{C}(g)$ is a Dwyer-Kan equivalence of simplicial categories.

We are now ready to prove

Theorem 1.17. *The adjoint pair*

$$\begin{tikzcd} \mathsf{sSet} \arrow[r, bend left, "\mathfrak{C}"] \arrow[r, phantom, "\perp"] & \mathsf{SCat} \arrow[l, bend left, "N_{\mathrm{coh}}"] \end{tikzcd}$$

is a Quillen equivalence between the Joyal model category structure on simplicial sets and the Bergner model category structure on SCat.

Proof. Let us first show that $\mathfrak{C} \dashv N_{\mathrm{coh}}$ is Quillen pair. By Proposition 1.15, the right adjoint N_{coh} sends fibrations to fibrations. Thus, it suffices to show that, for every $n \in \mathbb{N}$, $\mathfrak{C}\partial\Delta[n] \to \mathfrak{C}\Delta[n]$ is a cofibration in SCat. If $n = 0$, $\mathfrak{C}\partial\Delta[n] \to \mathfrak{C}\Delta[n]$ is isomorphic to the map $\emptyset \to \mathbb{1}$, which is one of the generating cofibrations in SCat. For the case $n > 0$, if $\Delta[-1] := \emptyset$, there is a coequalizer diagram in sSet

$$\coprod_{0\le i<j\le n} \Delta[n-2] \rightrightarrows \coprod_{0\le i<j\le n} \Delta[n-1] \longrightarrow \partial\Delta[n]$$

We then obtain a coequalizer diagram in SCat

$$\coprod_{0\le i<j\le n} \mathfrak{C}\Delta[n-2] \rightrightarrows \coprod_{0\le i<j\le n} \mathfrak{C}\Delta[n-1] \longrightarrow \mathfrak{C}\partial\Delta[n]$$

By the way colimits are computed in SCat, we get that:

- $\mathrm{Ob}(\mathfrak{C}\partial\Delta[n]) = \mathrm{Ob}(\mathfrak{C}\Delta[n]) = [n]$;
- for every $0 \le i \le j \le n$, $\mathrm{Map}_{\mathfrak{C}\partial\Delta[n]}(i,j) = \mathrm{Map}_{\Delta[n]}(i,j)$, unless $i = 0$ and $j = n$, in which case $\mathrm{Map}_{\mathfrak{C}\partial\Delta[n]}(0,n)$ is isomorphic to the boundary of the cube $\Delta[1]^{n-1} \cong \mathrm{Map}_{\mathfrak{C}\Delta[n]}(0,n)$.

It follows that there is a pushout diagram in SCat:

$$\begin{tikzcd} U\partial(\Delta[1]^{n-1}) \arrow[r] \arrow[d] & \mathfrak{C}\partial\Delta[n] \arrow[d] \\ U\Delta[1]^{n-1} \arrow[r] & \mathfrak{C}\Delta[n] \end{tikzcd}$$

where U is the functor (2) and the top map picks the objects $0,\ n$ of $\mathfrak{C}\partial\Delta[n]$ and the isomorphism $\partial(\Delta[1]^{n-1}) \cong \mathrm{Map}_{\mathfrak{C}\partial\Delta[n]}(0,n)$. Since $U\partial(\Delta[1]^{n-1}) \to U\Delta[1]^{n-1}$ is in the saturation of the generating cofibration of SCat (see Lemma 1.13), so is $\mathfrak{C}\partial\Delta[n] \to \mathfrak{C}\Delta[n]$.

For every $K \in \mathsf{sSet}$ and every fibrant $\mathbb{C} \in \mathsf{SCat}$, if $f\colon K \to N_{\mathrm{coh}}\mathbb{C}$, its adjoint $f^\sharp\colon \mathfrak{C}K \to \mathbb{C}$ is the composite $\epsilon_{\mathbb{C}} \circ \mathfrak{C}(f)$ and $\epsilon_{\mathbb{C}}$ is a DK equivalence by Theorem 1.14. By Proposition 1.16, f is a Joyal equivalence if and only if $\mathfrak{C}(f)$ is a DK equivalence. Thus, $\mathfrak{C} \dashv N_{\mathrm{coh}}$ is a Quillen equivalence. □

2. Relative categories: the minimal model

In a model category $\mathcal{M}$, weak equivalences are often enough to describe homotopical information and constructions, but cofibrations and fibrations are useful in practice. For example:

(1) the homotopy category $\mathrm{Ho}(\mathcal{M}) = \mathcal{M}[\mathcal{W}^{-1}]$ only depends upon the underlying category and the weak equivalences of $\mathcal{M}$, but cofibrations and fibrations allow a better description of its hom-sets;
(2) as shown in [DHKS04], homotopy (co)limits can be defined in an essentially unique way using just the weak equivalences in $\mathcal{M}$. However, the most common models of homotopy (co)limits (via the Bousfield-Kan formula or the derived functor approach) use the rest of the structure of $\mathcal{M}$ as well.

The idea that specifying weak equivalences should suffice to determine a homotopy theory is made concrete by *relative categories*. A relative category $(\mathcal{C}, \mathcal{W})$ is a minimal presentation for the categorical localization $\mathcal{C}[\mathcal{W}^{-1}]$, but the *hammock localization* functor also provides mapping spaces for it ([DK80]). Moreover, the category RelCat of small relative categories admits a model category structure which is Quillen equivalent to the complete Segal space model category structure on $\mathsf{s}^2\mathsf{Set}$ ([BK12]). Hence:

(1) RelCat presents the homotopy theory of $(\infty,1)-$categories.
(2) Being a relative category (with weak equivalences given by the ones coming from the model category structure), RelCat itself is a homotopy theory for (the homotopy theory of) $(\infty,1)$-categories.

Thus, relative categories provide a natural *internal model*[2] for the homotopy theory of $(\infty,1)$-categories.

2.1. **Generalities and Hammock Localization.** Let us start by recalling what we mean by a relative category.

Definition 2.1.

(1) A *relative category* is a pair $(\mathcal{C}, \mathcal{W})$, where $\mathcal{C}$ is a category and $\mathcal{W}$ is a wide subcategory of $\mathcal{C}$[3]. Maps in $\mathcal{W}$ are called *weak equivalences* of $(\mathcal{C}, \mathcal{W})$.
(2) A *relative functor* from the relative category $(\mathcal{C}, \mathcal{W})$ to the relative category $(\mathcal{C}', \mathcal{W}')$ is a functor $f\colon \mathcal{C} \to \mathcal{C}'$ such that $f(\mathcal{W}) \subseteq \mathcal{W}'$.

[2] Up to size issues.
[3] This means that $\mathcal{W}$ is a subcategory of $\mathcal{C}$ containing all the objects of $\mathcal{C}$.

Small relative categories and relative functors form a category, denoted by RelCat. There are adjoints

$$\begin{tikzcd} \mathsf{Cat} \arrow[rr, "\mathrm{und}" description] & & \mathsf{RelCat} \arrow[ll, bend right, swap, "\min"] \arrow[ll, bend left, "\max"] \end{tikzcd} \tag{4}$$

Here, for $(\mathcal{C}, \mathcal{W}) \in \mathsf{RelCat}$ and $\mathcal{C} \in \mathsf{Cat}$:

- $\mathrm{und}(\mathcal{C}, \mathcal{W}) := \mathcal{C}$, the *underlying category* to $(\mathcal{C}, \mathcal{W})$;
- $\min(\mathcal{C}) := (\mathcal{C}, \mathrm{Ob}(\mathcal{C}))$, the *minimal relative category* associated to $\mathcal{C}$;
- $\max(\mathcal{C}) := (\mathcal{C}, \mathcal{C})$, the *maximal relative category* associated to $\mathcal{C}$.

Definition 2.2. The *homotopy category* of $(\mathcal{C}, \mathcal{W}) \in \mathsf{RelCat}$ is the category $\mathrm{Ho}(\mathcal{C}, \mathcal{W})$ given by the localization $\mathcal{C}[\mathcal{W}^{-1}]$.

Remark 2.3. For $\mathcal{C} \in \mathsf{Cat}$, $\mathrm{Ho}(\min(\mathcal{C})) \cong \mathcal{C}$, whereas $\mathrm{Ho}(\max(\mathcal{C})) \cong 1$, the terminal category.

To each relative category $(\mathcal{C}, \mathcal{W})$, we can associate a whole simplicial category, thus providing mapping spaces for $(\mathcal{C}, \mathcal{W})$.

Proposition 2.4 ([DK80], Prop 3.1 & Prop 3.3)**.** *There is a functor*

$$L^H \colon \mathsf{RelCat} \to \mathsf{SCat} \tag{5}$$

called the hammock localization functor *having the following properties, for every small relative category* $(\mathcal{C}, \mathcal{W})$*:*

(1) there is an equivalence of categories

$$\pi_0(L^H(\mathcal{C}, \mathcal{W})) \simeq \mathrm{Ho}(\mathcal{C}, \mathcal{W}),$$

where $\pi_0(L^H(\mathcal{C}, \mathcal{W}))$ is the category of components of $L^H(\mathcal{C}, \mathcal{W})$ (see Definition 1.4);

(2) if $f \colon X \to Y$ is a map in $\mathcal{C}$, then, for all $Z \in \mathrm{Ob}(\mathcal{C})$, there are induced maps

$$f_* \colon \mathrm{Map}_{L^H(\mathcal{C},\mathcal{W})}(Z, X) \to \mathrm{Map}_{L^H(\mathcal{C},\mathcal{W})}(Z, Y)$$

and

$$f^* \colon \mathrm{Map}_{L^H(\mathcal{C},\mathcal{W})}(Y, Z) \to \mathrm{Map}_{L^H(\mathcal{C},\mathcal{W})}(X, Z),$$

which are Kan-Quillen equivalences if f is in $\mathcal{W}$.

An explicit construction of L^H was given in Alex's talk and can be found in [DK80], Section 2.1.

2.2. **The induced model structure from CSs.** Consider the functor

$$K \colon \Delta \times \Delta \to \mathsf{RelCat}, \quad ([m], [n]) \mapsto \min([m]) \times \max([n]).$$

Extending it along the Yoneda embedding $\mathbf{y} \colon \Delta \times \Delta \to \mathsf{s}^2\mathsf{Set}$ gives an adjoint pair

$$\begin{tikzcd} \mathsf{s}^2\mathsf{Set} \arrow[rr, bend left, "K = \mathrm{Lan}_{\mathbf{y}}(K)"] \arrow[rr, phantom, "\perp"] & & \mathsf{RelCat} \arrow[ll, bend left, "N_{\mathrm{Rzk}}"] \end{tikzcd} \tag{6}$$

where N_{Rzk} is the classifying diagram functor. Thus, the (m, n)-bisimplices of $N_{\mathrm{Rzk}}(\mathcal{C}, \mathcal{W})$ can be described as the relative functors

$$\min([m]) \times \max([n]) \to (\mathcal{C}, \mathcal{W}).$$

Analogously to what happens for the Thomason model structure on Cat (see [Tho80]), it turns out that, in order to transfer the complete Segal space model structure to a Quillen equivalent one on RelCat, one needs to modify the above adjunction. For this purpose, we introduce some operations on *relative posets*, i.e. relative categories $(\mathcal{C}, \mathcal{W})$ in which $\mathcal{C}$ is a poset.

Definition 2.5. Let $(\mathcal{P}, \mathcal{W})$ be a relative poset.

(1) The *terminal subdivision* of $(\mathcal{P}, \mathcal{W})$ is the relative poset $\xi_t(\mathcal{P}, \mathcal{W})$ defined as follows:
- the underlying poset of $\xi_t(\mathcal{P}, \mathcal{W})$ has elements given by the poset monomorphisms $[n] \to \mathcal{P}$ (for every $n \in \mathbb{N}$) and, for two such monomorphisms $x_1 \colon [n_1] \to \mathcal{P}$ and $x_2 \colon [n_2] \to \mathcal{P}$,
$$x_1 \leq x_2 \iff \exists\, s \colon [n_1] \to [n_2]\ (x_2 s = x_1)^4;$$
- $x_1 \leq x_2$ is a weak equivalence in $\xi_t(\mathcal{P}, \mathcal{W})$ if and only if $x_1(n_1) \leq x_2(n_2)$ is a weak equivalence in $(\mathcal{P}, \mathcal{W})$.

(2) The *initial subdivision* of $(\mathcal{P}, \mathcal{W})$ is the relative poset $\xi_i(\mathcal{P}, \mathcal{W})$ defined as follows:
- the underlying poset of $\xi_i(\mathcal{P}, \mathcal{W})$ is the opposite of the underlying poset of $\xi_t(\mathcal{P}, \mathcal{W})$;
- $x_2 \leq x_1$ is a weak equivalence in $\xi_i(\mathcal{P}, \mathcal{W})$ if and only if $x_2(0) \leq x_1(0)$ is a weak equivalence in $(\mathcal{P}, \mathcal{W})$.

Remark 2.6. The underlying poset of $\xi_t(\mathcal{P}, \mathcal{W})$ can also be described as having elements given by $(n+1)$-tuples $(x_0, \ldots, x_n)$ of elements of $\mathcal{P}$ such that $x_0 < x_1 < \cdots < x_n$, with $(x_0, \ldots, x_n) \leq (y_0, \ldots, y_m)$ if and only if $n \leq m$ and every x_i is (exactly) one of the y_j's.

Given a relative poset $(\mathcal{P}, \mathcal{W})$, there are relative functors

$$\pi_t \colon \xi_t(\mathcal{P}, \mathcal{W}) \to (\mathcal{P}, \mathcal{W}) \quad \text{and} \quad \pi_i \colon \xi_i(\mathcal{P}, \mathcal{W}) \to (\mathcal{P}, \mathcal{W}) \tag{7}$$

given by evaluating an element $x \colon [n] \to \mathcal{P}$ at n and at 0 respectively.

Let RelPos be the full subcategory of RelCat spanned by the relative posets. Then the terminal and the initial subdivision extend to endofunctors

$$\xi_t, \xi_i \colon \mathsf{RelPos} \to \mathsf{RelPos}$$

For a map $f \colon (\mathcal{P}, \mathcal{W}) \to (\mathcal{P}', \mathcal{W}')$ of relative posets and a poset monomorphism $x \colon [n] \to \mathcal{P}$, $\xi_t f(x) = \xi_i f(x)$ is the monomorphism in the epi-mono factorization of $f \circ x$. The projection functors π_t, π_i become natural transformations

$$\pi_t \colon \xi_t \to \mathrm{Id}_{\mathsf{RelPos}} \quad \text{and} \quad \pi_i \colon \xi_i \to \mathrm{Id}_{\mathsf{RelPos}}$$

Definition 2.7. The *two-fold subdivision* functor is

$$\xi := \xi_t \xi_i \colon \mathsf{RelPos} \to \mathsf{RelPos}$$

Set also

$$\pi := \pi_i \circ (\pi_t)_{\xi_i} \colon \xi \to \mathrm{Id}_{\mathsf{RelPos}} \tag{8}$$

We use ξ to modify the adjunction (6). Consider the functor

$$K_\xi \colon \Delta \times \Delta \to \Delta, \quad ([m],[n]) \mapsto \xi(\min([m]) \times \max([n]))$$

We obtain the adjoint pair

$$\begin{tikzcd} \mathsf{s^2Set} \arrow[rr, bend left, "K_\xi = \mathrm{Lan}_{\mathbf{y}}(K_\xi)"] \arrow[rr, phantom, "\perp"] & & \mathsf{RelCat} \arrow[ll, bend left, "N_\xi"] \end{tikzcd} \tag{9}$$

[4] Note that if such an s exists, it must be unique, because x_2 is a monomorphism.

For $(\mathcal{C}, \mathcal{W}) \in \mathsf{RelCat}$ and $m, n \in \mathbb{N}$, we have

$$N_\xi(\mathcal{C}, \mathcal{W})_{m,n} = \mathsf{RelCat}(\xi(\min([m]) \times \min([n])), (\mathcal{C}, \mathcal{W})).$$

We can now use this adjunction to transfer the model structure CSs to RelCat.

Theorem 2.8 ([BK12], Thm 6.1)**.** *There is a cofibrantly generated, left proper model category structure on* RelCat *in which a relative functor* $f\colon (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$ *is a weak equivalence (resp. a fibration) if and only if* $N_\xi(f)$ *is a weak equivalence (resp. a fibration) in* CSs*. Furthermore, every cofibrant object is a relative poset.*

So, in this model structure, every relative category (say, $(\mathsf{Kan}, \mathcal{W}_{\mathrm{KQ}})$ with $\mathcal{W}_{\mathrm{KQ}}$ being the class of Kan-Quillen equivalences) is weakly equivalent to a poset!

We call the model category structure of Theorem 2.8 the *Barwick-Kan model structure* on RelCat.

From Theorem 2.8 it is immediate that $K_\xi \dashv N_\xi$ is a Quillen pair $\mathsf{CSs} \to \mathsf{RelCat}$.

Theorem 2.9 ([BK12], Prop 10.3 & Thm 6.1)**.** *There is a Quillen equivalence*

$$\begin{tikzcd} \mathsf{CSs} \arrow[r, bend left, "K_\xi"] \arrow[r, phantom, "\perp"] & \mathsf{RelCat} \arrow[l, bend left, "N_\xi"] \end{tikzcd}$$

where RelCat *has the Barwick-Kan model structure.*

Therefore, the Barwick-Kan model structure turns RelCat into a presentation of $(\infty, 1)$-categories.

The functor N_ξ is needed to detect fibrations in RelCat. However, there are other ways to decide whether a relative functor is a weak equivalence. The natural transformation $\pi\colon \xi \to \mathrm{Id}_{\mathsf{RelPos}}$ of (8) gives rise to a natural transformation

$$\pi^*\colon N_{\mathrm{Rzk}} \to N_\xi \tag{10}$$

Proposition 2.10 ([BK12], Lemma 5.4)**.** *The natural transformation* π^* *of* (10) *is a pointwise vertical weak equivalence of bisimplicial set.*

The above Proposition gives the equivalence between the first two assertions in the following

Theorem 2.11 ([BK10], Thm 1.4)**.** *The following are equivalent, for a map* f *in* RelCat*.*

(1) f *is a weak equivalence in the Barwick-Kan model structure.*
(2) $N_{\mathrm{Rzk}}(f)$ *is a weak equivalence in* CSs*.*
(3) $L^H(f)$ *is a Dwyer-Kan equivlaence of simplicial categories.*

We conclude by stating the following result which ensures that fibrant objects in the Barwick-Kan model structure include many of the better-behaved presentations of homotopy theory.

Theorem 2.12 ([Mei15], Thm 4.12)**.** *The underlying relative category of every fibration category is a fibrant object in the Barwick-Kan model structure in* RelCat*.*

Conclusion

In these two talks we surveyed various presentations of $(\infty, 1)-$categories as model categories and we exhibited Quillen equivalences among some of those. By taking the direction of a Quillen equivalence to be given by the right adjoint, we can picture these equivalences as follows

$$
\begin{tikzcd}
\mathsf{SCat} \arrow[r, bend left, "N_{\mathrm{coh}}"] & (\mathsf{sSet})_{\mathrm{Joyal}} \arrow[r, bend left, "t^!"] & \mathsf{CSs} \arrow[l, bend left, "i_1^*"] & \mathsf{RelCat} \arrow[l, bend right, "N_\xi"']
\end{tikzcd}
\tag{11}
$$

Since right Quillen functors compose (as well as Quillen equivalences), we managed to find a functor $R\colon \mathsf{HMod} \to \mathsf{QCat}$ as stated in the Introduction, for $\mathsf{HMod} = \mathsf{CSs}$, SCat and RelCat. If we do not insist in only having (right) Quillen functors in our picture, we can expand (11) as follows

$$
\begin{tikzcd}
\mathsf{SCat} \arrow[r, bend left, "N_{\mathrm{coh}}"] & (\mathsf{sSet})_{\mathrm{Joyal}} \arrow[r, bend left, "t^!"] & \mathsf{CSs} \arrow[l, bend left, "i_1^*"] & \mathsf{RelCat} \arrow[l, bend right, "N_\xi"'] \arrow[l, bend left, "N_{\mathrm{Rzk}}"] \arrow[lll, bend right=60, "L^H"']
\end{tikzcd}
\tag{12}
$$

In this case, all the displayed functors are relative functors between relative categories. In fact,they are *DK-equivalences* of relative categories[5], in the sense that they are relative functors $F\colon (\mathcal{C}, \mathcal{W}) \to (\mathcal{C}', \mathcal{W}')$ which induce equivalences of homotopy categories and Kan-Quillen equivalences between the mapping spaces in the hammock localizations of $\mathcal{C}$ and $\mathcal{C}'$. Thus, all these functors (up to taking derived functors for the right Quillen maps) provide a suitable notion of "equivalence between homotopy theories", again in the spirit of the Introduction.

[5] Up to size issues.

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