1 Introduction

Let $S$ be a finite monoid and let $a \in S$. Define

\[ H_n(a) = \{(x_1, \ldots, x_n) \in S^n \mid x_1 \cdot \ldots \cdot x_n = a\} \]

and let

\[ h_n(a) = |H_n(a)|. \]

To get a grip on the behaviour of $\{h_n(a) \mid n \geq 1\}$ we define

\[ h(a) = \sum_{n \geq 1} h_n(a)t^{n-1}. \]

$h(a)$ was studied in [2] from a combinatorial perspective. We found that it could be pictured as a generalization of the generating function for the Zeta Polynomial $Z(P, n)$ of a finite poset $(P, \geq)$ [3; §3.11].

In this paper we continue our study of $\{h(a) \mid a \in S\}$. But this time we approach it from a more algebraic perspective. For example

(i) What is the recurrence formula for $\{h_1(a), h_2(a), \ldots\}$ and how are these recurrence formulae (as $a \in S$ varies) related? (See Proposition 2.4.)

(ii) Is there a canonical collection $\{R_0, R_1, \ldots, R_{n-1}\}$ of rational functions that forms a basis of $\text{Span}_Q\{h(a) \mid a \in S\}$? (See Proposition 2.2.)

(iii) What role does $\mathbb{Q}[S]$ (the monoid algebra) play in the discussion? (See Section 2.)

We consider the example $S = M_n(F_q)$ in detail. In a future paper we shall use the results of this paper to obtain explicit calculations for a large class of finite monoids.
2 The Element $\alpha$

Let $S$ be a finite monoid and let $\mathbb{Q}[S]$ be the monoid algebra of $S$ over $\mathbb{Q}$. Define

$$\alpha = \sum_{s \in S} s \in \mathbb{Q}[S].$$

2.1 Lemma. (i) $\alpha^n = \sum_{s \in S} h_n(s)s$ where $h_n(s) = |\{(x_1, \ldots, x_n) \in S^n \mid x_1 \cdot \ldots \cdot x_n = a\}|$.

(ii) $\frac{\alpha}{1 - \alpha t} = \sum_{s \in S} \left( \sum_{n \geq 1} h_n(s)t^{n-1} \right) s =: \sum_{s \in S} h(s)s$ where $h(s) \in \mathbb{Q}(t)$.

Proof. (i) results from gathering the terms of a straightforward calculation. Everything in (ii) is a formality except the fact that $h(s) \in \mathbb{Q}(t)$. But that follows from [2; Theorem 2.2].

A well-known theorem of undergraduate algebra says that we can write

$$\alpha^n = A_0 + A_1\alpha + \cdots + A_{n-1}\alpha^{n-1}$$

where $\{1, \alpha, \ldots, \alpha^{n-1}\} \subseteq \mathbb{Q}[S]$ is linearly independent, and $\{A_i\} \subseteq \mathbb{Q}$.

Furthermore, the $A_i$’s are unique. We write

$$\min(\alpha) = X^n - A_{n-1}X^{n-1} - \cdots - A_1X - A_0.$$  

2.2 Proposition. $\frac{\alpha}{1 - \alpha t} = R_0 + R_1\alpha + \cdots + R_{n-1}\alpha^{n-1}$ where

$$R_0 = A_0\frac{t^{n-1}}{D},$$

$$R_{n-1} = \frac{t^{n-2}}{D},$$

$$R_{n-2} = \frac{t^{n-3}}{D} - A_{n-1} \frac{t^{n-2}}{D},$$

$$\vdots$$

$$R_2 = \frac{t}{D} - A_{n-1} \frac{t^2}{D} - \cdots - A_3 \frac{t^{n-2}}{D},$$

$$R_1 = \frac{1}{D} - A_{n-1} \frac{t}{D} - \cdots - A_2 \frac{t^{n-2}}{D},$$

and

$$D = 1 - t(A_0t^{n-1} + \cdots + A_{n-1}).$$

Furthermore, $A_0 = 0$ if $S$ has more than one invertible element. So $R_0 = 0$ in that case.
Proof. Write \( \frac{\alpha}{1-\alpha t} = R_0 + R_1 \alpha + \cdots + R_{n-1} \alpha^{n-1} \). So \( \alpha = (1-\alpha t)(R_1 + R_1 \alpha + \cdots + R_{n-1} \alpha^{n-1}) \). Expand this out using \( \alpha^n = A_0 + A_1 \alpha + \cdots + A_{n-1} \alpha^{n-1} \), and we obtain the advertised formulae for \( R_i \), \( i = 0, 1, \ldots, n-1 \).

To complete the proof we only need to show that \( A_0 = 0 \) if \( G(S) = \{ s \in S \mid sg = gs = 1 \} \) for some \( g \in S \).

If \( s \in G(S) \), one checks easily that \( h_n(s) = g^n \) where \( g = |G(S)| \). So \( h(s) = \frac{1}{1-g^t} \). Now let \( \alpha = \sum_{s \in S} s \in Q[S] \) and let \( \overline{\alpha} \in Q[S]/\langle S \setminus G(S) \rangle \cong Q[G] \). With this identification

\[
\overline{\alpha} = \sum_{s \in G(S)} s,
\]

so that \( \min(\overline{\alpha}) = X^2 - gX \) as long as \( g > 1 \). But if \( f \in Q[x] \) and \( f(\alpha) = 0 \) then \( f(\overline{\alpha}) = 0 \). So \( \min(\overline{\alpha}) | \min(\alpha) \). But \( X | \min(\overline{\alpha}) \), and so the constant term of \( \min(\alpha) \) is zero. \( \square \)

2.3 Remark. (a) A simple calculation shows that

\[
\text{Span}_Q\{R_0, \ldots, R_{n-1}\} = \begin{cases} \text{Span}_Q \left\{ \frac{1}{D}, \frac{t}{D}, \ldots, \frac{t^{n-2}}{D} \right\} & \text{if } A_0 = 0 \\ \text{Span} \left\{ \frac{1}{D}, \frac{t}{D}, \ldots, \frac{t^{n-1}}{D} \right\} & \text{if } A_0 \neq 0 \end{cases}
\]

In particular, \( \{R_1, \ldots, R_{n-1}\} \) is linearly independent over \( Q \) (and if \( A_0 \neq 0 \) then \( \{R_0, \ldots, R_{n-1}\} \) is linearly independent).

(b) An easy calculation shows that if we can write

\[
\frac{\alpha}{1-\alpha t} = \sum_{i=1}^m h_i(t)x_i
\]

where \( \{x_i\} \subseteq Q[S] \) is a \( Q \)-basis, and \( \{h_i(t)\} \subseteq Q(t) \), then

\[
\text{Span}_Q\{h_i(t)\}_{i=1}^m = \text{Span}_Q\{R_i\}_{i=1}^{n-1}.
\]

It follows that \( (D) = \{g \in Q[t] \mid gh(a) \in Q[t]\} \).

(c) We can also express the \( A_i \)'s in terms of the \( R_i \)'s. As before, expand out

\[
\alpha = (1-\alpha t)(R_0 + R_1 \alpha + \cdots + R_{n-1} \alpha^{n-1})
\]

and again use \( \alpha^n = A_0 + A_1 \alpha + \cdots + A_{n-1} \alpha^{n-1} \). We then obtain

\[
\alpha^n = \frac{R_0}{tR_{n-1}} + \left( \frac{R_1 - 1 - tR_0}{tR_{n-1}} \right) \alpha + \left( \frac{R_2 - tR_1}{tR_{n-1}} \right) \alpha^2 + \cdots + \left( \frac{R_{n-1} - tR_{n-2}}{tR_{n-1}} \right) \alpha^{n-1}.
\]
Thus,

\[
\begin{align*}
A_0 &= \frac{R_0}{tR_{n-1}} \\
A_1 &= \frac{R_1 - tR_0}{tR_{n-1}} \\
A_2 &= \frac{R_2 - tR_1}{tR_{n-1}} \\
& \vdots \\
A_{n-1} &= \frac{R_{n-1} - tR_{n-2}}{tR_{n-1}}
\end{align*}
\]

because \(\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\} \subseteq \mathbb{Q}(t)[S]\) is linearly dependent over \(\mathbb{Q}(t)\).

2.4 Proposition. Let \(f(X) = X^d - (B_0 + B_1X + \cdots + B_{d-1}X^{d-1})\). Then the following are equivalent, where \(\alpha = \sum_{s \in S} s\).

\(i\) \(f(\alpha) = 0\).

\(ii\) \(\min(\alpha) \mid f\).

\(iii\) \(h_{d+m}(s) = B_0h_m(s) + B_1h_{m+1}(s) + \cdots + B_{d-1}h_{m+d-1}(s)\) for all \(m \geq 0\). Here we use the convention

\[
h_0(s) = \begin{cases} 
1 & \text{if } s = 1 \\
0 & \text{if } s \neq 1
\end{cases}
\]

\(iv\) \(a\) \(h(s) t^d f(1/t) \in \mathbb{Q}[t] \) for any \(s \in S\).

\(b\) If \(t^k \mid \min(\alpha)\) then \(t^k \mid f\).

Proof. \(i\) and \(ii\) are equivalent by elementary algebra.

\(iii\) is equivalent to saying

\[\sum_{s \in S} h_d(s)s = B_0 + B_1 \left( \sum_{s \in S} h_1(s)s \right) + \cdots + B_{d-1} \left( \sum_{s \in S} h_{d-1}(s)s \right)\]

so that (using \(\sum_{s \in S} h_\ell(s)s = \alpha^\ell\))

\[\alpha^d = B_0 + B_1\alpha + \cdots + B_{d-1}\alpha^{d-1}\].

Hence, \(i\) and \(iii\) are equivalent.

Before we consider \(iv\) notice that if \(g(t) = t^k \prod_{i=1}^s (t - \alpha_i)^{n_i}\) then \(t^{\deg(g)} g(1/t) = \prod_{i=1}^s (1 - \alpha_it)^{n_i}\). It follows that \(g(t) \mid f(t)\) if and only if \(t^{\deg(g)} g(1/t) \mid t^d f(1/t)\), and \(t^k \mid f\) whenever \(t^k \mid g\).

But from 2.2, \(D = t^{\min(\alpha)}(1/t)\), while from Remark 2.3 \(b\), \(h(s)t^d f(1/t) \in \mathbb{Q}[t]\) if and only if \(D \mid t^d f(1/t)\). Thus, \(iv\) is equivalent to \(ii\).
2.5 Theorem. With \( \{ R_i(t) \mid i = 0, \ldots, n-1 \} \) as in 2.2 we obtain
\[
h(a) = h_0(a)R_0(t) + h_1(a)R_1(t) + \cdots + h_{n-1}(a)R_{n-1}(t) .
\]

Proof. Let \( h_i = h_i(a), h = h(a) \). Then
\[
h =: \sum_{i \geq 1} h_i t^{i-1}
\]
\[
= h_1 + h_2 t + \cdots + h_{n-1} t^{n-2}
+ (\epsilon + A_1 h_1 + \cdots + A_{n-1} h_{n-1}) t^{n-1}
+ (A_0 h_1 + \cdots + A_{n-1} h_n) t^n
+ (A_0 h_2 + \cdots + A_{n-1} h_{n+1}) t^{n+1}
+ \cdots
\]
(\text{using 2.4 with } f = \min(\alpha), \text{ where } \epsilon = A_0 \text{ if } a = 1 \text{ and } \epsilon = 0 \text{ if } a \neq 1)
\[
= h_1 + h_2 t + \cdots + h_{n-1} t^{n-2}
+ A_0 t^n h + \delta
+ A_1 t^{n-1} h
+ A_2 t^{n-2}(h - h_1)
+ \cdots
+ A_{n-1} t(h - (h_1 + h_2 + \cdots + h_{n-2}))
\]
(where \( \delta = A_0 t^{n-1} \) if \( a = 1 \) and zero otherwise).

Thus, gathering terms
\[
h(1 - t(A_0 t^{n-1} + \cdots + t A_{n-2} + A_{n-1})) = h_1 + h_2 t + \cdots + h_{n-1} t^{n-2} + \delta
- A_2 t^{n-2} h_1
- A_3 t^{n-3}(h_1 + h_2 t)
+ \cdots
- A_{n-1} t(h_1 + h_2 t + \cdots + h_{n-2} t^{n-3})
= h_1 + (h_2 - A_{n-1} h_1) t + (h_3 - A_{n-1} h_2 - A_{n-2} h_1) t^2
+ \cdots + (h_{n-1} - A_{n-1} h_{n-2} - \cdots - A_2 h_1) t^{n-2} + \delta
= h_1(1 - A_{n-1} t - \cdots - A_2 t^{n-2})
+ h_2(t - A_{n-1} t^2 - \cdots - A_3 t^{n-2})
+ \cdots
+ h_{n-1} t^{n-2} + \delta .
\]

The conclusion follows immediately from 2.2. \( \square \)
2.6 Remark. In practice, one should calculate $h(a)$ using a slight variation of Theorem 2.5. Indeed, for $a \in S$, let

$$\Omega_a = \{x \in S \mid a \notin SxS\}.$$ 

Then $\Omega_a$ is the two-sided ideal of elements that cannot be involved in any product resulting in $a$. Thus we can replace $S$ by the Rees quotient $S/\Omega_a$ [1; page 17]. Usually, the minimal polynomial of $\alpha$ will have lower degree for $S/\Omega_a$ than for $S$.

3 How To Find $\text{Min}(\alpha)$

The results of 2.4 and 2.5 imply that, if we can compute $\text{min}(\alpha)$, everything else will fall into place; once we calculate $\{h_1(a), \ldots, h_{n-1}(a) \mid a \in S\}$. In this section we consider the problem of actually finding $\text{min}(\alpha)$ in terms of salient properties of $S$. We first recall some results from [2].

Let $S$ be a finite monoid and let $a \in S$. Define

$$\overline{a} = \{b \in S \mid aS = bS\} = \{a_1, \ldots, a_s\}.$$ 

Define

$$R_{\overline{a}} = \begin{pmatrix} r(a_1/a_1) & r(a_2/a_1) & \ldots \\ r(a_1/a_2) & r(a_2/a_2) & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

where

$$r(b/a) = |\{x \in S \mid a = bx\}|.$$ 

We say $b \geq a$ if $a = bx$ for some $x \in S$. By [2; Theorem 2.2] we have

$$h(a_i) = \frac{1}{\det(I_s - tR)} \left( P_{a_i} + t \sum_{b > a} P(b/a_i)h(b) \right) \quad (*)$$

where

$$\begin{pmatrix} P_{a_1} \\ \vdots \\ P_{a_s} \end{pmatrix} = \text{Adj}(I_s - tR) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} P(b/a_1) \\ \vdots \\ P(b/a_s) \end{pmatrix} = \text{Adj}(I_s - tR) \begin{pmatrix} r(b/a_1) \\ \vdots \\ r(b/a_s) \end{pmatrix}$$

The following result gives us a crude starting point in our pursuit of $\text{min}(\alpha)$.
3.1 Proposition. Let \( S/R = S/\sim \), where \( a \sim b \) if \( aS = bS \). For \( \pi \in S/R \) define \( R_\pi \) as above. Let \( \alpha = \sum_{s \in S} s \). Then \( \min(\alpha) \) divides \( t^k(\prod_{\pi \in S/R} \det(tI_s - R_\pi)) \) for some \( k \geq 0 \).

Proof. By the above formula (*), if \( D \) is the generator of \( \{ f \in \mathbb{Q}[t] \mid fh(a) \in \mathbb{Q}[t] \text{ for all } a \in S \} < \mathbb{Q}[t] \). Then \( D \) divides \( f(t) = \prod_{\pi \in S/R} \det(I_s - tR_\pi) \). Thus, if \( \deg(D) = m \) and \( \deg(f) = n \) then \( t^mD(1/t) \) divides \( t^n f(1/t) \). But \( \min(\alpha) = t^k t^m D(1/t) \) for some \( k \geq 0 \) by remark 2.3.2 and the proof of 2.2. Thus, \( \min(\alpha) \) divides \( t^{k+n} f(1/t) = t^k(\prod_{\pi \in S/R} \det(tI_s - R_\pi)) \). \( \square \)

A finite monoid \( S \) is \( \mathcal{R} \)-homogeneous [2] if

(i) \( a_1 \sim a_2 \sim a_3 \sim a_4 \) implies \( r(a_1/a_2) = r(a_3/a_4) \)

(ii) \( a_1 \sim a_2 < b \) implies \( r(b/a_1) = r(b/a_2) \).

By [2; Theorem 5.2] we have the following result:

Let \( S \) be \( \mathcal{R} \)-homogeneous and let \( a \in S \). Then

\[
h(a) = \frac{1}{1 - saa^t} \left( 1 + t \sum_{b > a} r(b/a) h(b) \right)
\]

(**) where \( r_a = r(a/a) \) and \( s_a = |\{ x \in S \mid x \sim a \}| \).

3.2 Proposition. Let \( S \) be \( \mathcal{R} \)-homogeneous.

(i) The following are equivalent.

(a) \( \min(\alpha)(0) \neq 0 \).

(b) \( \{ s \in S \mid st = 1 \text{ for some } t \in S \} =: G(S) = \{ 1 \} \).

(ii) If \( G(S) \nsubseteq \{ 1 \} \) then \( X|\min(\alpha) \) get \( X^2 \nmid \min(\alpha) \).

Proof. From the proof of 2.2 we see that \( h(s) = \frac{1}{1-gt} \) for \( s \in G(S) \), where \( g = |G(S)| \). We also showed that if \( S = G(S) \), then

\[
\min(\alpha) = \begin{cases} X^2 - gX & \text{if } |G(S)| = g > 1 \\ X - 1 & \text{if } |G(S)| = 1 \end{cases}
\]

So we proceed inductively using 2.4. Let \( a \in S \setminus G(S) \). So \( a < 1 \). Assume inductively that \( f(X) = X^n - (B_0 + B_1 X + \cdots + B_{n-1} X^{n-1}) \), and assume further that

(i) \( h_n(b) = B_0 h_0(b) + B_1 h_1(b) + \cdots + B_{n-1} h_{n-1}(b) \) for any \( b > a \).

(ii) \( X^2 \nmid f \).
It follows from (**) above that
\[ h_{m+1}(a) - r h_m(a) = \sum_{b > a} r(b/a) h_m(b) \]
for any \( m \geq 0 \), where \( r = s_a r_a \)

since we have adopted the convention that
\[ h_0(a) = \begin{cases} 1 & \text{if } a = 1 \\ 0 & \text{if } a \neq 1 \end{cases} \]

So we calculate
\[
\begin{align*}
h_{m+1}(a) - r h_m(a) &= \sum_{b \succ a} r(b/a) [h_0(b) + \cdots + B_{n-1} h_{n-1}(b)] \\
&= \sum_{i=0}^{n-1} B_i \left( \sum_{b \succ a} r(b/a) h_i(b) \right) \\
&= \sum_{i=0}^{n-1} B_i [h_{i+1}(a) - r h_i(a)].
\end{align*}
\]

Thus, \( h(a) \) “satisfies” the polynomial
\[ g(X) = (X - r)(X^n - (B_0 + B_1 X + \cdots + B_{n-1} X^{n-1})). \]

But if \( b > a \) then \( h(b) \) “satisfies” this polynomial because \( f(X) \) is a factor.

Inductively, this procedure manufactures a polynomial
\[ h(X) = (X - r_n)(X - r_{n-1}) \cdots (X - r_1) f_0(X) \]

where
\[ f_0(X) = \begin{cases} X^2 - gX & \text{if } |G(S)| = g > 1 \\ X - 1 & \text{if } |G(S)| = 1 \end{cases} \]

whose coefficients provide a linear recurrence for \( \{ h(a) \mid a \in S \} \) in the sense of Proposition 2.4 (iii). It follows from 2.4 that \( \min(\alpha)|h(x) \). The other conclusions follow from this and our formula for \( h(X) \).

\[ \square \]

3.3 Proposition. Let \( S \) be \( R \)-homogeneous and let \( I(S) = \{ r \in \mathbb{N} \mid r = s_a r_a \text{ for some } a \in S \} \). Then
\[ \min(\alpha) = \begin{cases} X \prod_{r \in I(S)} (X - r)^{n_r} & \text{if } |G(S)| > 1 \\ \prod_{r \in I(S)} (X - r)^{n_r} & \text{if } |G(S)| = 1 \end{cases} \]

where \( 1 \leq n_r \leq \max \left\{ m \mid s_{a_0} r_{a_1} = r \text{ for some chain } a_0 < a_1 < \cdots < a_n \right\} \)
Proof. All that remains here is to prove the inequalities involving \( \{ n_r \mid r \in I(S) \} \). The upper bound amounts to showing that for any \( a \in S \) the pole of
\[
h(a) = \frac{1}{1 - r_a s_a t} \left( 1 + t \sum_{b > a} r(b/a) h(b) \right)
\]
at \( 1/r \) has order less than or equal to
\[
n_r(a) = \max \left\{ n \mid s_a r_a = r \text{ for some chain } \ a_0 < a_1 < \cdots < a_n \right\}
\]
But this follows immediately (by induction) once we assume it is true for \( h(b), b > a \).

To see that each \( n_r > 0 \), consider \( L : \mathbb{Q}[S] \rightarrow \mathbb{Q}[S] \) defined by \( L_a(a) = a \alpha \). Clearly, \( \min(\alpha) = \min(L_a) \).

So we compute. Let \( a \in S \). Then \( a \alpha = a \sum s = \sum_{e \sim a} r(a/e) e + \sum_{x < a} m_x x \) where \( r(a/a) = |\{ s|as = e \}| \) and \( m_x \) is some nonnegative integer. But \( S \) is \( R \)-homogeneous, and so \( r(a/e) = r_a = r_e \). Thus,
\[
a \alpha = a_r \left( \sum_{e \sim a} e \right) + \sum_{x < a} m_x x.
\]
So let \( A = \sum_{e \sim a} e \in \mathbb{Q}[S] \). Then
\[
A \alpha = r_a s_a A + \sum_{x < a} \ell_x x
\]
where \( s_a = |\{ e \in S \mid e \sim a \}| \) and \( \ell_x \) is some nonnegative integer.

Notice that \( V = \text{Span}_\mathbb{Q}\{x|x < a\} \) is \( L_a \)-stable. Thus, \( L_a \) has an eigenvector in \( \mathbb{Q} \cdot A \oplus V \subseteq \mathbb{Q}[S] \) with the eigenvalue \( r_a s_a \). Hence \( X - r_a s_a \) is a factor of \( \min(\alpha) \).

3.4 Corollary. (a) Suppose \( S \) is \( R \)-homogeneous. Then the following are equivalent.

(i) \( \alpha \) is semisimple.

(ii) \( \min(\alpha) = \begin{cases} X \prod_{i=1}^{n} (X - r_i), & |G(S)| > 1 \\
\prod_{i=1}^{n} (X - r_i), & |G(S)| = 1 \end{cases} \) where \( \{ r_i \} = \{ s_a r_a \mid a \in S \} \).

(iii) For any \( a \in S \), \( h(a) \) has simple poles.

(b) Furthermore, if \( S \) has the property “\( a \leq b \) and \( r_a s_a = r_b s_b \) together imply that \( a \sim b \)” then \( \alpha \) is semisimple.
3.0.1 Example

In this example we consider the example $S = M_n(F_q)$ in detail, where $F_q$ is the field with $q$ elements.

Let $a \in S$. Then for some $g, h \in G(S)$,

$$gah = e_i = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 0 & & & & & & \\ & & & 1 & & & & & \\ & & & & 0 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 0 & & \\ & & & & & & & \cdots & \\ \\ & & & & & & & & 1 \\ \end{pmatrix}$$

for some $0 \leq i \leq a$ (where $i = \text{rank } (e_i)$). It follows easily that

$$r(a/a) = r(e_i/e_i) = \left| \{ x \in S \mid e_i = e_iX \} \right| .$$

An elementary calculation verifies

$$r_i = r(e_i/e_i) = q^{n(n-i)} .$$

Next we need to find

$$s_i = \left| \{ b \in S \mid b \sim e_i \} \right| = \left| e_iG \ell_n(F_q) \right| .$$

Another simple calculation yields

$$s_i = (q^n - 1)(q^n - q) \cdot \ldots \cdot (q^n - q^{n-1}) .$$

So

$$s_i r_i = q^{n(n-i)}(q^n - 1)(q^n - q) \cdot \ldots \cdot (q^n - q^{n-1}) .$$

Notice that $s_i r_i \neq s_j r_j$ if $i \neq j$. Thus by 3.4 above, $\alpha$ is semisimple, and so

$$\min(\alpha) = X \prod_{i=0}^{n}(X - q^{n(n-i)}(q^n - 1) \cdot \ldots \cdot (q^n - q^{n-1})) \quad \text{and}$$

$$D = \prod_{i=0}^{n}(1 - q^{n(n-i)}(q^n - 1) \cdot \ldots \cdot (q^n - q^{n-1})X) .$$

Furthermore, by (**) of section 3

$$h(e_i) = \frac{1}{1 - r_i s_i t} \left( 1 + t \sum_{b > e_i} r(b/e_i)h(b) \right) .$$

We use this to find an explicit formula of the form

$$h(e_i) = \frac{1}{1 - r_i s_i t} \left( 1 + t \sum_{j>i} A_i h(e_i) \right) .$$
We first calculate
\[ r(b/e_i) = \left| \{ x \in S \mid bx = e_i \} \right| . \]

Suppose \( bx = e_i \). Then
\[ \{ x \mid bx = e_i \} = x + \{ y \mid by = 0 \} . \]
Thus \( r(b/e_i) = q^d \) where \( d = \dim_{F_q}(\{ y \mid by = 0 \}) \). An elementary calculation shows that \( d = n(n - j) \), where \( j = \text{rank}(b) \). To find the sought after formulae, it remains to find
\[
\left\{ b \in S \mid \begin{array}{l}
bx = e_i \text{ for some } x \\
\text{rank } (b) = j
\end{array} \right\} \text{ for each } j > i .
\]

Define
\[ X_j = \left\{ b \in S \mid \begin{array}{l}
bx = e_i \text{ for some } x \\
\text{and rank } (b) = j
\end{array} \right\} \text{ for each } j > i .
\]
Define
\[ P_i = \{ g \in G \mid ge_i = e, ge_i \} \]
where
\[ G = G\ell_n(F_q) . \]
Notice also that
\[ P_i X_j G = X_j \text{ for all } j > i . \]

After a little calculation we conclude the following:

Let
\[ \mathcal{R}_n = \left\{ A \in M_n(F_q) \mid \begin{array}{l}A \text{ is a 01 matrix with at} \\
\text{most one nonzero entry in}
\end{array} \right. \]
\[ \text{each row or column} \left. \right\} . \]
and let
\[ X_j = \{ r \in \mathcal{R}_n \mid e_i \in r\mathcal{R}_n, \text{ rank } (r) = j \} . \]
Then
\[ X_j = P_i X_j G . \]
Now
\[ \mathcal{X} = \bigcup_{j>i} X_j = \bigcup_{e \in E(\mathcal{X})} eW \]
and after a little more calculation we obtain
\[ \mathcal{X} = C_W(e_i) \Lambda(\mathcal{X}) W \]
where \( W \subseteq \mathcal{R}_n \) is the unit group and
\[ \Lambda(\mathcal{X}) = \{ e_{i+1}, e_{i+2}, \ldots, e_n = 1 \} . \]
Thus,
\[ X_j = P_i e_j G . \]
and

\[ X = \bigcup_{j=i+1}^{n} X_j. \]

But

\[ |X_j| = (q^j - 1) \cdots (q^j - q^{j-1}) \left[ \begin{array}{c} u \\ j \end{array} \right]_q \left[ \begin{array}{c} n - i \\ j - i \end{array} \right]_q \]

where

\[ \left[ \begin{array}{c} n \\ j \end{array} \right]_q = \frac{(q^n - 1) \cdots (q - 1)}{[(q^j - 1) \cdots (q - 1)][(q^{n-k} - 1) \cdots (q - 1)]}. \]

So

\[ h(e_j) = \frac{1}{1 - r_is_it} \left( 1 + t \sum_{b>e_j} r(b/e_i)h(b) \right) \]

\[ = \frac{1}{1 - r_is_it} \left( 1 + t \sum_{j=i+1}^{n} \sum_{\substack{b>e_j \\ \text{rank}(b)=j}} r(b/e_i)h(b) \right) \]

\[ = \frac{1}{1 - r_is_it} \left( 1 + t \sum_{j=i+1}^{n} q^{n(n-j)}(q^j - 1) \cdots (q^j - q^{j-1}) \left[ \begin{array}{c} n \\ j \end{array} \right]_q \left[ \begin{array}{c} n - i \\ j - i \end{array} \right]_q h(e_j) \right) \]

\[ = \frac{1 + t \sum_{j=i+1}^{n} q^{n(n-j)}(q^j - 1) \cdots (q^j - q^{j-1}) \left[ \begin{array}{c} n \\ j \end{array} \right]_q \left[ \begin{array}{c} n - i \\ j - i \end{array} \right]_q h(e_j)}{1 - q^n(n-i)(q^n - 1) \cdots (q - 1)t}. \]

References

