Why $\infty! = \sqrt{2\pi}$

Masoud Khalkhali

Mathematics Department, UWO, Canada

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Abstract

A few years ago I gave a Pizza Seminar talk where I showed how to regularize a divergent infinite sum like $1 + 2 + 3 + 4 + 5 + \cdots$ and get -1/12. In this talk I shall discuss a multiplicative version and show how one can regularize infinite products like $1.2.3.4. \cdots$ and get a finite number. This topic is intimately related to Stirling's formula, and to Riemann's zeta function, its analytic continuation , functional equation, and special values. Some tools of classical analysis like Euler-Maclaurin summation formula will be introduced and used.

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- But imagine we want to regularize this infinity and get a finite number. How would you proceed? For example we want to know how fast these numbers $n!$ grow. But how fast with respect to what? Can we throw away a divergent bad part and keep a finite convergent component?

First approach: Stirling's formula

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- Stirling's formula

$$n! = \sqrt{2\pi}\sqrt{n}\left(\frac{n}{e}\right)^n\left(1 + O\left(\frac{1}{n}\right)\right).$$

It shows how fast $\log n!$ grows compared to some standard functions like $n^\alpha (\log n)^\beta$:

$$\log n! = n \log n + \frac{1}{2}\log n - n + \log\sqrt{2\pi} + O(\frac{1}{n})$$

Regularizing $\infty!$

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$$\log \infty! = \log \sqrt{2\pi}$$

- Equivalently $\quad \lim_{n\to\infty} \frac{n!}{\sqrt{n}\left(\frac{n}{e}\right)^n} = \sqrt{2\pi}.$

  So again we set $\quad \infty! = \sqrt{2\pi}.$

Second approach: zeta regularization

- The Riemann zeta function, originally defined as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \qquad \Re(s) > 1,$$

  is convergent (and holomorphic) only in $\Re(s) > 1$. But it has an analytic continuation to $\mathbb{C} \setminus \{1\}$, with a simple pole at $s = 1$.

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- A formal manipulation shows a way to regularize ∞! In fact

$$\zeta'(s) = \sum_{n=1}^{\infty} (n^{-s})' = \sum_{n=1}^{\infty} -\frac{\ln n}{n^s}$$

Second approach: zeta regularization

- Put $s = 0$ (this is illegal-why?), and get

$$\zeta'(0) = -\sum_{n=1}^{\infty} \log n = -\log(1 \cdot 2 \cdot 3 \cdots) = -\log(\infty!)$$

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$$\infty! = e^{-\zeta'(0)}$$

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- Note that our manipulations are wrong and illegal, but the final definition makes sense and gives a finite number! It is a mystery that this kind of regularization is so useful in mathematics and physics. Will it turn out to be the same number we got using Stirling's formula? Yes!

What is analytic continuation and how it is done in practice?

- Our original formula for $\zeta(s)$

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is divergent in the left half plane $\Re(s) \leq 1$, but it has an analytic continuation to $\mathbb{C} \setminus \{1\}$. How is this possible?

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  is divergent in the left half plane $\Re(s) \leq 1$, but it has an analytic continuation to $\mathbb{C} \setminus \{1\}$. How is this possible?

- Standard method to find an analytic continuation of an analytic function $f(z)$: Find a different formula for $f(z)$ which is manifestly defined and holomorpic on a larger domain.

Analytic continuation of $\zeta(s)$

- A simple example: $f(z) = \sum_{n=0}^{\infty} z^n$ is only convergent and defined for $|z| < 1$, but a different formula for it, $f(z) = \frac{1}{1-z}$, is clearly analytic in $\mathbb{C} \setminus \{1\}$.

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- We apply the same idea to Riemann zeta function. Is there a different formula for $\zeta(s)$ that is manifestly analytic in a larger domain? Yes, and in fact there are many formulas and all of them are rather hard to find. In his 1859 magnificient paper, Riemann gave at least two other formulas for $\zeta(s)$ that leads to its analytic continuation. Here we give yet another formula that is based on Euler-Maclaurin summation formula. We need to know about Bernoulli numbers first.

Enter Bernoulli numbers

- Bernoulli numbers $B_m, m = 0, 1, 2, \cdots$ are defined by their generating function:

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}$$

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- Easy to see that

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_{2n+1} = 0, \quad n = 1, 2, 3,$$

These numbers are ubiquitous: they appear in analysis, geometry, topology, and numerical analysis.

Euler-Maclaurin Summation

- This formula turns summation into integration and vice-versa, with a remainder term that can be effectively computed/estimated:

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(x)\,dx + \sum_{k=1}^{n} \frac{B_k}{k!} \left(f^{(k-1)}(b) - f^{(k-1)}(a) \right) + R_n.$$

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- A heuristic proof: look for a function g s.t.

$$g(x+1) - g(x) = f(x)$$

  Then

$$f(a)+f(a+1)+\cdots+f(b-1) = g(a+1)-g(a)+\cdots+g(b)-g(b-1)$$

$$= g(b) - g(a)$$

Euler-Maclaurin Summation

- How to find this g? Let $D = \frac{d}{dx}$. Taylor's formula gives:

$$f(x) = g(x+1) - g(x) = (\sum \frac{D^n}{n!})g(x) = (e^D - 1)g(x)$$

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- Rewrite it as

$$g(x) = \frac{D}{e^D - 1}h(x), \qquad Dh(x) = f(x), \quad h(x) = \int_a^x f(t)dt.$$

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- Solution (Bernoulli numbers appear!)

$$g(x) = (\sum_{n=0}^{\infty} B_n \frac{D^n}{n!})h(x).$$

Notice that $g(b) - g(a) = h(b) - h(a) = \int_a^b f(x)dx$.

Bernoulli polynomials

Bernoulli polynomials $B_k(x), k = 0, 1, 2 \cdots$ are defined recursively:

$$B_0(x) = 1$$

$$B_n'(x) = nB_{n-1}(x) \quad \text{and} \quad \int_0^1 B_n(x)\,dx = 0 \text{ for } n \geq 1$$

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Here are the first few

$$B_0(x) = 1$$

$$B_1(x) = x - 1/2$$

$$B_2(x) = x^2 - x + 1/6$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$$

Bernoulli polynomials

- Alternatively, they can be defined by their generating function

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.$$

In particular

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$$\bar{B}_n(x) = B_n\left(x - \lfloor x \rfloor\right)$$

Remainder term for EM summation

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- Example: for $n = 1$, we get

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(x)\,dx - \frac{1}{2}\left(f(b) - f(a)\right)\right) + \int_a^b f'(x)(x - [x])dx$$

Proof: Integration by parts!

Example: for $n = 2$, we get

$$\sum_{k=a}^{b-1} f(k) = \int_a^b f(x)\,dx - \frac{1}{2}\left(f(b) - f(a)\right) + \frac{1}{12}\left(f'(b) - f'(a)\right)$$

$$-\frac{1}{2}\int_a^b f''(x)\bar{B}_2(x)dx$$

Faulhaber-Bernoulli formulae

- As a first application we get a formula for power sums. Let $f(x) = x^p$. Then the remainder $R_n = 0$ for $n > p$ and we get an exact formula

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=0}^{p} (-1)^j \binom{p+1}{j} B_j n^{p+1-j}.$$

$$= \frac{1}{p+1} n^{p+1} + \frac{1}{2} n^p + \frac{p}{2} B_2 n^{p-1} + \cdots .$$

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- Example:

$$1^5 + 2^5 + 3^5 + \cdots + n^5 = \frac{2n^6 + 6n^5 + 5n^4 - n^2}{12}$$

A new formula for $\zeta(s)$

- As a second application of Euler-Maclaurin summation we obtain a new formula for $\zeta(s)$ that is manifestly extendible to larger domains. Let $f(x) = x^{-s}$. We get, for $s \neq 1$,

$$\sum_{m=1}^{N} \frac{1}{m^s} = \frac{1 - N^{1-s}}{s-1} + \frac{1 + N^{-s}}{2}$$

$$+ \sum_{k=2}^{n} B_k s(s+1) \cdots (s+k-2)(1 - N^{-s-k+1})/k! + R_n$$

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- Let $N \to \infty$ with $Re(s) > 1$ fixed. We get

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{k=2}^{n} B_k s(s+1)\cdots(s+k-2)/k!$$

$$-\frac{1}{n!} s(s+1)\cdots(s+n-1) \int_1^{\infty} \bar{B}_n(x) x^{-s-n} dx$$

- Example: for $n = 1$, we get

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s\int_1^{\infty} (x - [x] - 1/2)x^{-s-1}dx.$$

The integral is convergent if $Re(s) > 0$. This already extends zeta to the larger domain $Re(s) > 0$ (with a simple pole at s=1).

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- In general, since $\bar{B}_n(x)$ is periodic, it is bounded on $[1, \infty]$ and hence $\int_1^\infty \bar{B}_n(x)x^{-s-n}dx$ is convergent if $Re(s) > 1 - n$.

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- Thus the new expression for $\zeta(s)$ shows that it can be analytically continued to the larger domain $Re(s) > 1 - n$. By choosing larger and larger values of n we see that $\zeta(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$.

Special zeta values

The formula that we obtained before,

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The formula

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \frac{1}{12}s - \frac{1}{2}s(s+1)\int_1^\infty \bar{B}_2(x)x^{-s-2}dx,$$

shows that

$$\zeta(0) = -\frac{1}{2}.$$

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$$\zeta(-1) = -\frac{1}{12}$$

And in general we get

$$\zeta(-m) = -\frac{B_{m+1}}{m+1}, \qquad m \geq 0$$

Special zeta values

These are usually written in a mystifying way:

$$
\begin{aligned}
1+1+1+\cdots &= -\frac{1}{2} \\
1+2+3+\cdots &= -\frac{1}{12} \\
1^2+2^2+3^2+\cdots &= 0 \\
1^3+2^3+3^3+\cdots &= \frac{1}{120} \\
&\cdots\cdots \\
1^m+2^m+3^m+\cdots &= -\frac{B_{m+1}}{m+1}
\end{aligned}
$$

These formulas were obtained by Euler in 18th century. His interpretation of the sums were different though.

Derivative of zeta at $s = 0$

- Recal the regularization scheme we are using:

$$1 \times 2 \times 3 \times \cdots = e^{-\zeta'(0)}$$

Calculating $\zeta'(0)$ is much harder! I don't know of any derivation that does not use the functional equation for zeta. So let me recall it.

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- First we need to know about Euler's constant γ, and his Gamma function $\Gamma(s)$. The first is defined through Taylor expansion of $(s-1)\zeta(s)$ at $s = 1$

$$(s-1)\zeta(s) = 1 + \gamma(s-1) + \cdots$$

Equivalently,

$$\gamma = (\log(s-1)\zeta(s))'|_{s=1}$$

The Gamma function

- It is an analytic extension of the factorial function $n \mapsto (n-1)!$ defined by
$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}, \quad Re(s) > 0.$$

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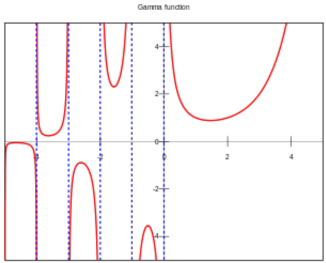
- It is easy to see, using integration by parts, that $\Gamma(n) = (n-1)!$ and $\Gamma(s+1) = s\Gamma(s)$. The latter relation in turn implies that $\Gamma(s)$ has a meromorphic extension to $\mathbb{C}$ with simple poles at $s = 0, -1, -2, -3, \cdots$.

In the course of computing $\zeta'(0)$, we need the following two formulas for Gamma, known as reflection formula and duplication formula:

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},$$

$$\Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

And here is a graph of $\Gamma(s)$ for real s.



The functional equation

- This is the relation
$$Z(s) = Z(1-s)$$
where $Z(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$ is the completed zeta function. Assuming this, we can proceed as follows. The functional equation can be written as
$$(s-1)\zeta(s) = -2^s\pi^{s-1}\ \sin\left(\frac{\pi s}{2}\right)\ \Gamma(2-s)\ \zeta(1-s)$$

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$$(s-1)\zeta(s) = -2^s\pi^{s-1}\ \sin\left(\frac{\pi s}{2}\right)\ \Gamma(2-s)\ \zeta(1-s)$$

- Log differentiate this at $s = 1$. We get
$$\gamma = -\frac{\Gamma'(1)}{\Gamma(1)} + \log 2\pi - \frac{\zeta'(0)}{\zeta(0)}$$
$$= \gamma + \log 2\pi - \frac{\zeta'(0)}{\zeta(0)}$$

So:

$$\frac{\zeta'(0)}{\zeta(0)} = \log 2\pi.$$

Since we already know that $\zeta(0) = -\frac{1}{2}$, we obtain

$$\zeta'(0) = -\frac{1}{2}\log 2\pi,$$

or,

$$1 \times 2 \times 3 \times \cdots = e^{-\zeta'(0)} = \sqrt{2\pi}.$$

Summary

- We sketched two approaches to regularize divergent infinite products like $1 \times 2 \times 3 \times \cdots$: via Stirling's formula and via the zeta faunction. The zeta regularization needed more sophisticated tools: analytic continuation (which was done thanks to Euler-Maclaurin summation formula), and evaluation of $\zeta'(0)$ which used the functional equation for $\zeta(s)$.

Summary

- We sketched two approaches to regularize divergent infinite products like $1 \times 2 \times 3 \times \cdots$: via Stirling's formula and via the zeta fauncion. The zeta regularization needed more sophisticated tools: analytic continuation (which was done thanks to Euler-Maclaurin summation formula), and evaluation of $\zeta'(0)$ which used the functional equation for $\zeta(s)$.

- The zeta regularization has the advantage of being systematic and can be applied in far more general situations to regularize a divergent infinite product like $\prod \lambda_i$. For a sequence

$$\Lambda : \quad 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \quad \lambda_n \to \infty,$$

Summary-continued

- If the series
$$\zeta_\Lambda(s) = \sum \frac{1}{\lambda_i^s},$$
is convergent (hence analytic) for $Re(s)$ large enough, and if it has analytic continuation and is regular at $s = 0$, we can define
$$\prod \lambda_i := e^{-\zeta_\Lambda'(0)}$$
- Is regularization a useful concept? Absolutely! Determinant of Laplacians, analytic torsion, regularization in quantum field theory and the Casimir effect, are a few examples of its vast applications in mathematics and physics.


Pizza
Stirling's
numbers
Seminar
infinite
function
Riemann
1+2+3+4+5+
sum
divergent
1.2.3.4.5
summation
In
formula
values
like
special
1/12
regularize
zeta
continuation
regularization
Euler-Maclaurin
Eulerequation.
products
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Riemann's
Bernoulli