Localization Theory in Triangulated Categories: a (brief, incomplete and probably superficial) overview.

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1 Localizing categories and localization functors.

Localization of categories is quite a ubiquitous framework throughout Mathematics. One should always be careful in making such assertive statements, but in the specific case a (well-known) example may suffice as a justification. Namely, let $\mathscr{C}$ be a category and let $i\colon \mathscr{D} \longrightarrow \mathscr{C}$ be the inclusion of a reflective (replete) subcategory of $\mathscr{C}$ with left adjoint $a\colon \mathscr{C} \longrightarrow \mathscr{D}$. Then $(\mathscr{D},\ a)$ is a localization of $\mathscr{C}$ with respect to the class $\mathcal{W}$ of all those morphisms f in $\mathscr{C}$ such that $a(f)$ is invertible in $\mathscr{D}$. Furthermore, $\mathcal{W}$ admits a calculus of left fractions and $\mathscr{D}$ is the category of $\mathcal{W}-$local objects (see below for the definitions). In particular, every locally presentable category, being equivalent to a reflective subcategory of a category of presheaves, is a localization (see [GaZi] and [AdRo]).

Localization theory addresses the following problem. Given a category $\mathscr{C}$, one often has a class $\mathcal{W}$ of certain arrows (sometimes called *weak equivalences*) which are not necessarily invertible, but one would like to consider them to be isomorphisms. This means of course that one seeks for a category $\mathscr{C}[\mathcal{W}^{-1}]$, called *localization of* $\mathscr{C}$ *at* $\mathcal{W}$ and for a *quotient functor* $\mathscr{C} \longrightarrow \mathscr{C}[\mathcal{W}^{-1}]$ which sends each arrow in $\mathcal{W}$ into an isomorphism and is universal among all those functors doing so. Such an inverting problem is formally always solvable, as one can construct $\mathscr{C}[\mathcal{W}^{-1}]$ (up to equivalences) as a category having the same objects of $\mathscr{C}$ and arrows given by certain equivalence classes of paths made by morphisms in $\mathscr{C}$ and formal inverses of morphisms in $\mathcal{W}$. Here the adverb "formally" is required as the category $\mathscr{C}[\mathcal{W}^{-1}]$ is not, in general, locally small, that is, given $X, Y \in \mathscr{C}[\mathcal{W}^{-1}]$, $\mathrm{Hom}_{\mathscr{C}[\mathcal{W}^{-1}]}(X,Y)$ needs not to be a small set, unless $\mathscr{C}$ is itself a small category (see [Kr1], Example 4.15). However, this set-theoretical issue (which is usually ignored) is not the only cause of concern: in general, the description of arrows in $\mathscr{C}[\mathcal{W}^{-1}]$ is far from being handy and even deciding whether $\mathrm{Hom}_{\mathscr{C}[\mathcal{W}^{-1}]}(X,Y)$ consists of a single element may be problematic.

A way to overcome these problems is well known to algebraic topologists (and to homotopical algebraists). Indeed, if $\mathcal{W}$ is the class of weak equivalences in a *model category* $\mathscr{M}$ with classes of fibrations and cofibrations given by $\mathrm{Fib}(\mathscr{M})$ and $\mathrm{Cof}(\mathscr{M})$ respectively, then $\mathscr{M}[\mathcal{W}^{-1}]$ is usually called the *homotopy category* of $\mathscr{M}$ and denoted by $\mathrm{Ho}(\mathscr{M})$. In this case, for each $X, Y \in \mathscr{M}$, $\mathrm{Ho}(\mathscr{M})(X,Y)$ is isomorphic to the small set

$$\mathscr{M}(QRX, QRY)/\sim,$$

where Q and R are the (canonical) cofibrant and the fibrant replacement functors respectively and $\sim$ is the homotopy relations between maps in $\mathscr{M}$ (see [Hov] for definitions and results). Moreover, the class $\mathcal{W}$ of weak equivalences is *saturated*, i.e. a map $f\colon X \to Y$ in $\mathscr{M}$ is a weak equivalence (i.e. is in $\mathcal{W}$) precisely when $[f] \in \mathrm{Ho}(\mathscr{M})$ is an isomorphism. Just to mention an example, if $\mathscr{A}$ is an abelian category with enough injectives, then the category $\mathbf{Ch}^{\geq 0}(\mathscr{A})$ of non-negatively graded cochain complexes in $\mathscr{A}$ has a model category structure where weak equivalences are given by quasi-isomorphisms and fibrations are epimorphisms of cochain complexes having injective kernels in each degree. The homotopy category of $\mathbf{Ch}^{\geq 0}(\mathscr{A})$ with respect to this model structure is the full subcategory $\mathbf{D}^{\geq 0}(\mathscr{A})$ of the *derived category* $\mathbf{D}(\mathscr{A})$ of $\mathscr{A}$ given by non-negatively graded cochain complexes.

In a general context, a somewhat nicer description of $\mathscr{C}[\mathcal{W}^{-1}]$ can be provided when $\mathcal{W}$ *admits a calculus of left* (or *right*) *fractions*. This means essentially that $\mathcal{W}$ contains the identity maps of all objects in $\mathscr{C}$, is closed under composition of its elements and satisfies a couple of technical properties known as *left* (or *right*) *Ore condition* and *left* (or *right*) *cancellability.* When $\mathcal{W}$ admits a calculus of left fractions, a morphism from X to Y in $\mathscr{C}[\mathcal{W}^{-1}]$ has a representative of the form

$$X \xrightarrow{\alpha} Y' \xleftarrow{\sigma} Y \ ,$$

where α is an arrow in $\mathscr{C}$ and $\sigma \in \mathcal{W}$. Subclasses of morphisms in a category $\mathscr{C}$ admitting a calculus of left fractions arise naturally in presence of a *localization functor* on $\mathscr{C}$, i.e. of an endofunctor L of $\mathscr{C}$ for which there exists a natural transformation $\eta\colon \mathrm{Id}_{\mathscr{C}} \longrightarrow L$ such that $L\eta$ is invertible and $L\eta = \eta L$. Localization functors on $\mathscr{C}$ are strictly connected with reflective subcategories of $\mathscr{C}$. Indeed, an endofunctor $L\colon \mathscr{C} \longrightarrow \mathscr{C}$ is a localization functor exactly when there are a functor $F\colon \mathscr{C} \longrightarrow \mathscr{D}$ and a fully faithful right adjoint $G\colon \mathscr{D} \longrightarrow \mathscr{C}$ such that $L = G \circ F$. Furthermore, if $\mathcal{W}(L)$ denotes the class of morphisms α in $\mathscr{C}$ such that $L(\alpha)$ is invertible, then $\mathcal{W}(L) = \mathcal{W}(F)$ admits a calculus of left fractions and $\mathscr{C}[\mathcal{W}(L)^{-1}]$ is identified with $\mathscr{D}$ which is, in turn, equivalent to the *essential image* of L, $\mathrm{Im}(L)$, made precisely of all *L-local* objects, that is of all $Z \in \mathscr{C}$ such that, whenever $\sigma \in \mathcal{W}$, $\mathscr{C}(\sigma, Z)$ is an isomorphism.

2 Triangulated categories and their localizations.

Classes of weak equivalences $\mathcal{W}$ admitting a calculus of left and right fractions are called *multiplicative systems.* They play a central role when dealing with localization in the context of triangulated categories. A *triangulated category* is a couple $(\mathscr{T}, \Sigma)$, where $\mathscr{T}$ is an additive category and Σ is a self-equivalence of $\mathscr{T}$ (called *suspension functor*), together with a class of *distinguished triangles* (or *exact triangles*) of the form

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X \ , \tag{1}$$

satisfying a set of coherent, well-known axioms (see [Nee01]). A way to look at these axioms is provided as follows. Given a triangle (1) in a category $\mathscr{C}$, we refer to α as the *homotopy kernel* of β and to β as the *homotopy cokernel* of α. Then the axioms for a pre-triangulated category loosely correspond to the following statements (together with closure under isomorphisms of triangles for the class of distinguished triangles):

Tr0. The identity has homotopy kernel and cokernel equal to zero.

Tr1. Every morphism has a homotopy kernel and cokernel.

Tr2. Any morphism is the homotopy kernel of its homotopy cokernel (up to sign), and any morphism is the homotopy cokernel of its homotopy kernel (up to sign).

Tr3. Homotopy kernels and cokernels are weakly functorial.

Employing the notion of a *homotopy cartesian square*, one can also reinterpret the fifth axiom **Tr4.** (usually known as the *"ochtaedral axiom"*) for a triangulated category in "homotopical" terms. An explanation for this point of view can be given considering the quintessential example of a triangulated category, that is, the *homotopy category* $\mathbf{K}(\mathscr{A})$ *of (complexes on) an abelian category* $\mathscr{A}$. Indeed, such an additive category admits a suspension functor given by the shifting functor $\Sigma\colon \mathbf{K}(\mathscr{A}) \longrightarrow \mathbf{K}(\mathscr{A})$ sending a complex X to the shifted complex $X[1]$. Then there is a triangulation on $(\mathbf{K}(\mathscr{A}), \Sigma)$ given by all triangles in $\mathbf{K}(\mathscr{A})$ isomorphic to the images in $\mathbf{K}(\mathscr{A})$ of sequences in $\mathbf{Ch}(\mathscr{A})$ of the form

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} C(\alpha) \xrightarrow{\gamma} \Sigma X \ ,$$

where α is a map of complexes and $C(\alpha)$ is the *mapping cone* of α. The reason why one ought to call $\beta\colon Y \to C(\alpha)$ the homotopy cokernel of α relies on the simple fact that a map of complexes $\delta\colon Y \to Z$ factors through β if and only if $\delta \circ \alpha$ is null-homotopic (i.e. if and only if the image of $\delta \circ \alpha$ in $\mathbf{K}(\mathscr{A})$ is the zero morphism). Similar considerations apply for the homotopy kernel of α.

Completely analogous factorization results carry over on every (pre)triangulated category, as one can show that homotopy (co)kernels are actually *weak (co)kernels* in $\mathscr{T}$ and are unique up to (non unique) isomorphism. Indeed, triangulated categories can be viewed somehow as a *nonlinear* (or *weakened*) *version* of abelian categories, where the role of (short) exact sequences is played by exact triangles. For example, if one calls an arrow α in a triangulated category $\mathscr{T}$ a *homotopy monomorphism* (*homonomorphism* for short) if its homotopy kernel is zero and a *homotopy epimorphism* (*hoepimorphism*) whenever its homotopy cokernel is zero, then one gets that α is an isomorphism exactly when it is both an homonomorphism and a hoepimorphism. This result represents the weakened version of the well-known characterization of isomorphisms in abelian categories as monomorphisms which are also epimorphisms. Actually, if we consider again our main example $\mathbf{K}(\mathscr{A})$, we see that, albeit being always a triangulated category, $\mathbf{K}(\mathscr{A})$ is rarely an abelian category; for example, if $\mathscr{A}$ is the category of (right) module over a ring R, then $\mathbf{K}(\mathscr{A})$ is abelian only if R is a semisimple ring. The same problem also arises for the *derived category* $\mathbf{D}(\mathscr{A})$ of $\mathscr{A}$, which is obtained from $\mathbf{Ch}(\mathscr{A})$ (or from $\mathbf{K}(\mathscr{A})$) by inverting quasi-isomorphisms: localizing at chain homotopy equivalences or at quasi-isomorphisms destroys the abelian structure on $\mathbf{Ch}(\mathscr{A})$.

Luckily, this bad behaviour with respect to localizations for the abelian structure does not persist in triangulated categories, at least to some reasonable extent. Indeed, let $\mathscr{T}$ be a triangulated category with suspension functor Σ and consider a multiplicative system $\mathcal{W}$ in $\mathscr{T}$ (so $\mathcal{W}$ admits a calculus of both left and right fractions) which is *compatible with the triangulation*, i.e. $\mathcal{W}$ is closed under iterated suspensions and desuspensions of morphisms ($\Sigma^n(\sigma) \in \mathcal{W}$ for all $\sigma \in \mathcal{W}$ and all $n \in \mathbb{Z}$) and for each triangle (1) with α, β in $\mathcal{W}$, there is a triangle $(\alpha,\ \beta,\ \gamma')$ with $\gamma' \in \mathcal{W}$. Under these hypotheses, the localization $\mathscr{T}[\mathcal{W}^{-1}]$ carries a unique triangulated structure such that the quotient functor $\mathscr{T} \longrightarrow \mathscr{T}[\mathcal{W}^{-1}]$ is an *exact functor* between triangulated categories (which essentially means that it sends exact triangles in $\mathscr{T}$ to exact triangles in $\mathscr{T}[\mathcal{W}^{-1}]$). In particular, the derived category $\mathbf{D}(\mathscr{A})$ of an abelian category $\mathscr{A}$ inherits a canonical triangulated structure from that of $\mathbf{K}(\mathscr{A})$, since quasi-isomorphisms in $\mathbf{K}(\mathscr{A})$ do form a multiplicative system which is compatible with the triangulation of $\mathbf{K}(\mathscr{A})$. The same paradigm shows that the following derived categories are triangulated (see [Kr2]):

- the derived category of a *dg-algebra*, which is a monoid object in the symmetric monoidal category of (co)chain complexes with monoidal structure given by the tensor product of complexes;
- the derived category of a category of modules over a (small) *dg-category*, i.e. a category enriched over chain complexes of modules;
- the derived category of an *exact category* (in the sense of Quillen), that is an additive category with a distinguished class of sequences which are called *exact* and satisfy suitable axioms.

Multiplicative systems compatible with triangulations abound in a triangulated category $(\mathscr{T},\ \Sigma)$ and arise in different manners:

(i) let $\mathscr{A}$ be an abelian category and $H\colon \mathscr{T} \longrightarrow \mathscr{A}$ a *cohomological functor* on $\mathscr{T}$, i.e. a functor sending every exact triangle in $\mathscr{T}$ to an exact sequence in $\mathscr{A}$ (for example, one can consider, for a fixed object $X \in \mathscr{T}$, the covariant Hom functor $\mathscr{T}(X, -)\colon \mathscr{T} \longrightarrow \mathbf{Ab}$). Then the class $\mathcal{W}$ of morphisms σ in $\mathscr{T}$ such that $H(\Sigma^n\sigma)$ is invertible for all $n \in \mathbb{Z}$ forms a multiplicative system which is compatible with the triangulation of $\mathscr{T}$;

(ii) let $\mathscr{S}$ be a *triangulated subcategory* of $\mathscr{T}$ (a non-empty, full and additive subcategory of $\mathscr{T}$ satisfying closure properties so that the collection of triangles having vertices in $\mathscr{S}$ and being distinguished in $\mathscr{T}$ gives a triangulation on $\mathscr{S}$). Set $\mathcal{W}(\mathscr{S})$ for the class of morphisms $X \to Y$ in $\mathscr{T}$ which fit into an exact triangle
$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X\ ,$$
where $Z \in \mathscr{S}$. Then $\mathcal{W}(\mathscr{S})$ is a multiplicative system which is compatible with the triangulation of $\mathscr{T}$. The localization $\mathscr{T}[\mathcal{W}(\mathscr{S})^{-1}]$ is denoted by $\mathscr{T}/\mathscr{S}$ and called the *(Verdier) localization* (or *quotient*) *of* $\mathscr{T}$ *by* $\mathscr{S}$. The quotient functor $Q\colon \mathscr{T} \longrightarrow \mathscr{T}/\mathscr{S}$ is the universal exact functor annihilating the objects of $\mathscr{S}$ and, furthermore, every cohomological functor annihilating $\mathscr{S}$ factors uniquely through Q;

(iii) given an exact localization functor $L\colon \mathscr{T} \longrightarrow \mathscr{T}$, its *kernel* is unsurprisingly the full subcategory of $\mathscr{T}$ consisting of all those objects $X \in \mathscr{T}$ such that $L(X) \cong 0$. Then $\mathrm{Ker}(L)$ is a *thick* triangulated subcategory of $\mathscr{T}$ (i.e. it is closed under retracts of its objects), so one can form the Verdier quotient as in (ii), for $\mathscr{S} := \mathrm{Ker}(L)$. Moreover, L induces an equivalence of categories $\mathscr{T}/\mathrm{Ker}(L) \simeq \mathrm{Im}(L)$.

Quotients like the one in (iii) above are called *Bousfield quotients* and the functor L is called a *Bousfield localization*. A natural question arises at this point: when is it the case that a thick subcategory $\mathscr{S}$ of $\mathscr{T}$ is the kernel of an exact localization functor? Equivalently, how do one characterize Bousfield quotients among Verdier quotients $\mathscr{T}/\mathscr{S}$ in terms of the quotienting subcategory $\mathscr{S}$ or of the quotient functor $Q\colon \mathscr{T} \longrightarrow \mathscr{T}/\mathscr{S}$? A first general answer is the following: a thick subcategory $\mathscr{S}$ of $\mathscr{T}$ is the kernel of a Bousfield localization exactly when $\mathscr{S}$ is coreflective in $\mathscr{T}$ or, equivalently, if and only if the quotient functor $\mathscr{T} \longrightarrow \mathscr{T}/\mathscr{S}$ has a right adjoint.

Strengthening a little bit the working hypotheses, one can find another solution to the above problem by means of the so-called *Brown representability* theorem. Namely, suppose that $\mathscr{T}$ is a triangulated category having small coproducts and assume that $\mathscr{T}$ is *perfectly generated* by a small set $\mathcal{S}$ of objects of $\mathscr{T}$. This means on the one hand that the smallest thick subcategory of $\mathscr{T}$ closed under coproducts (this kind of subcategories are called *localizing subcategories* of $\mathscr{T}$) and containing $\mathcal{S}$ is $\mathscr{T}$ itself, while, on the other hand, it provides a condition for a family of arrows $(X_i \to Y_i)_{i\in I}$ in $\mathscr{T}$ to be such that the induced map

$$\mathscr{T}\left(C,\ \coprod_{i\in I} X_i\right) \to \mathscr{T}\left(C,\ \coprod_{i\in I} Y_i\right)$$

is an epimorphism of abelian groups for all $C \in \mathcal{S}$. In these circumstances, Brown representability asserts that any cohomological functor $H\colon \mathscr{T}^{op} \longrightarrow \mathbf{Ab}$ preserving products is representable. As a consequence, one also gets an adjoint functor theorem: an exact functor $\mathscr{T} \longrightarrow \mathscr{U}$ between triangulated categories has a right adjoint if and only if it preserves small coproducts. Since the inclusion of a localizing subcategory of $\mathscr{T}$ is exact and preserves small coproducts, we then see that if $\mathscr{S}$ is a localizing subcategory of $\mathscr{T}$ which is also perfectly generated, then there is a localization functor $L\colon \mathscr{T} \longrightarrow \mathscr{T}$ such that $\mathscr{S} = \operatorname{Ker}(L)$. Nevertheless, Brown representability is a powerful tool in itself. It can be used, for example, to show the following result. Let $\mathscr{A}$ be an abelian category having an injective cogenerator and exact coproducts (this is the case when $\mathscr{A}$ is an abelian Grothendieck category, such as the category of sheaves of $\mathcal{O}_X-$modules on a ringed space $(X,\ \mathcal{O}_X)$), and denote by $\mathbf{K}_{\text{inj}}(\mathscr{A})$ the smallest full triangulated subcategory of $\mathbf{K}(\mathscr{A})$ which is closed under taking products and contains all injective objects of $\mathscr{A}$. Then the inclusion of $\mathbf{K}_{\text{inj}}(\mathscr{A})$ in $\mathbf{K}(\mathscr{A})$ has a left adjoint J such that, for all $X \in \mathbf{K}(\mathscr{A})$, the natural map $X \to JX$ is an injective resolution of X. Furthermore, precomposing the quotient functor $\mathbf{K}(\mathscr{A}) \longrightarrow \mathbf{D}(\mathscr{A})$ with the inclusion $\mathbf{K}_{\text{inj}}(\mathscr{A}) \longrightarrow \mathbf{K}(\mathscr{A})$ gives an equivalence of categories $\mathbf{K}_{\text{inj}}(\mathscr{A}) \simeq \mathbf{D}(\mathscr{A})$, so that, in particular, $\mathbf{D}(\mathscr{A})$ is a locally small category. Under this identification, J is a right adjoint for the quotient functor $\mathbf{K}(\mathscr{A}) \longrightarrow \mathbf{D}(\mathscr{A})$.

Suitable Bousfield localizations also provide collections of exact functors organized in triples of adjoint functors, known as recollements.

Definition 2.1. A *recollement* is a diagram of exact functors

$$\xymatrix{\mathscr{T}' \ar[r]|{i_*} & \mathscr{T} \ar@/_1.5pc/[l]_{i^*} \ar@/^1.5pc/[l]^{i^!} \ar[r]|{j^*} & \mathscr{T}'' \ar@/_1.5pc/[l]_{j_!} \ar@/^1.5pc/[l]^{j_*}} \tag{2}$$

such that:

(i) $(i^!,\ i_*)$, $(i_*,\ i^*)$, $(j_*,\ j^*)$ and $(j^*,\ j_!)$ are adjoint pairs (where left adjoints are on the left);

(ii) the units $\mathrm{Id} \to j^*j_!$, $\mathrm{Id} \to i^!i_*$ and the counits $j^*j_* \to \mathrm{Id}$, $i^*i_* \to \mathrm{Id}$ are isomorphisms;

(iii) $\operatorname{Im}(i_*) = \operatorname{Ker}(j^*)$.

Given a Bousfield localization functor $L\colon \mathscr{T} \longrightarrow \mathscr{T}$ such that $\operatorname{Ker}(L)$ is a reflective subcategory of $\mathscr{T}$, there is an induced recollement of the form (1), where $\mathscr{T}' := \operatorname{Ker}(L)$, $\mathscr{T}'' := \operatorname{Im}(L) \simeq \mathscr{T}/\operatorname{Ker}(L)$, i_* is the inclusion of $\operatorname{Ker}(L)$ into $\mathscr{T}$ and j^* is the corestriction of L to $\operatorname{Im}(L)$ or, under the identification $\operatorname{Im}(L) \simeq \mathscr{T}/\operatorname{Ker}(L)$, the quotient functor $\mathscr{T} \longrightarrow \mathscr{T}/\operatorname{Ker}(L)$. Actually, every recollement for $\mathscr{T}$ as in (1) can be recovered, up to equivalences, exactly in this way. Examples of recollements include the one induced on the homotopy category of a category of modules by the construction of the derived category and the recollement given on the category of modules over a ring by any idempotent of the ring itself (see [Kr1], §4.13).

3 Some vistas

The theory of (localization of) triangulated categories has several, widely spread applications. We collect here a couple of them.

Torsion pairs.

Definition 3.1. A *torsion pair* on a triangulated category $(\mathscr{T},\ \Sigma)$ is a pair $(\mathscr{X},\mathscr{Y})$ of full and replete subcategories of $\mathscr{T}$ satisfying the following conditions:

(i) $\mathscr{T}(\mathscr{X},\mathscr{Y})=0$ (meaning that, for all $X\in\mathscr{X}$ and all $Y\in\mathscr{Y}$, $\mathscr{T}(X,Y)=0$);

(ii) $\Sigma X\in\mathscr{X}$ and $\Sigma^{-1}Y\in\mathscr{Y}$ for each $X\in\mathscr{X}$ and $Y\in\mathscr{Y}$;

(iii) for each object $C\in\mathscr{T}$, there is a triangle

$$X_C\longrightarrow C\longrightarrow Y_C\longrightarrow \Sigma X_C \tag{3}$$

in $\mathscr{T}$ with $X_C\in\mathscr{X}$ and $Y_C\in\mathscr{Y}$

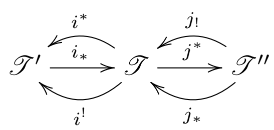
In this case $\mathscr{X}$ is called a *torsion class* and $\mathscr{Y}$ a *torsion free class.*

Given torsion pairs $(\mathscr{X},\mathscr{Y})$ and $(\mathscr{Y},\mathscr{Z})$, the triple $(\mathscr{X},\ \mathscr{Y},\ \mathscr{Z})$ is called a TTF (*torsion-torsion-free*) *triple.*

The promised connection between recollements and torsion pairs is given via TTF triples by the following

Proposition 3.2 ([Nic], §4.2)**.** *Let* $(\mathscr{T},\ \Sigma)$ *be a triangulated category.*

1. *For every recollement*



   as in (1)*,* $(\mathrm{Im}(j_!),\mathrm{Im}(i_*),\mathrm{Im}(j_*))$ *is a TTF triple in* $\mathscr{T}$*.*

2. *Assigning to each recollement* (1) *the TTF triple* $(\mathrm{Im}(j_!),\mathrm{Im}(i_*),\mathrm{Im}(j_*))$ *gives a bijection between equivalence classes of recollements and TTF triples in* $\mathscr{T}$*.*

Now, torsion pairs realise a bridge between the abelian and the triangulated world, through the usual mantra which associates to an abelian category its derived category. Indeed, one can define torsion pairs also in abelian categories, using short exact sequences to reformulate axiom (iii) of Definition 3.1 above. For example, in the abelian category $\mathbf{Mod}-\mathbb{Z}$ of abelian groups, the full and replete subcategories $\mathscr{S}$ and $\mathscr{G}$ of torsion and torsion-free abelian groups respectively are such that $(\mathscr{S},\ \mathscr{G})$ is a torsion pair. Moreover, given a pair $(\mathscr{S},\ \mathscr{G})$ of full and replete subcategories of an abelian category $\mathscr{A}$, one can define full and replete subcategories of $\mathbf{D}^b(\mathscr{A})$ (the bounded derived category of $\mathscr{A}$) by setting

$$\mathscr{X}(\mathscr{S}):=\{C^\bullet\in\mathbf{D}^b(\mathscr{A}):\ \forall n\in\mathbb{Z}_{>0}\ (H^n(C^\bullet)=0)\ \wedge\ H^0(C^\bullet)\in\mathscr{S}\}$$

and

$$\mathscr{Y}(\mathscr{G}):=\{C^\bullet\in\mathbf{D}^b(\mathscr{A}):\ \forall n\in\mathbb{Z}_{<0}\ (H^n(C^\bullet)=0)\ \wedge\ H^0(C^\bullet)\in\mathscr{S}\}.$$

Then one gets the following

Theorem 3.3 ([BeRe], §1.3)**.** *Let* $\mathscr{A}$ *be an abelian category and let* $(\mathscr{S},\ \mathscr{G})$ *be a pair of full and replete subcategories of* $\mathscr{A}$*. Then the following are equivalent:*

- $(\mathscr{S},\ \mathscr{G})$ *is a torsion pair in* $\mathscr{A}$*;*
- $(\mathscr{X}(\mathscr{S}),\ \mathscr{Y}(\mathscr{G}))$ *is a torsion pair in* $\mathbf{D}^b(\mathscr{A})$*.*

This theorem then also relates TTF triples in abelian categories to TTF triples in triangulated categories, hence to recollements and Bousfield localizations. In particular, if we consider a *tilting module* T over a finitely generated Artin algebra A (or, more generally, a tilting object in an abelian category), then we get a torsion pair in $\mathscr{A} := \mathbf{mod} - A$ (the category of finitely generated $A-$modules) given by $(\mathscr{S},\ \mathscr{G})$ where $\mathscr{S}$ is the full subcategory of $\mathscr{A}$ consisting of those finitely generated $A-$modules making $\mathrm{Ext}^1_A(T,-)$ vanish and, similarly, $\mathscr{G}$ is the full subcategory of $\mathscr{A}$ consisting of those $M \in \mathscr{A}$ such that $\mathrm{Hom}_A(T, M) = 0$. Thus we get a torsion pair $(\mathscr{X}(\mathscr{S}),\ \mathscr{Y}(\mathscr{G}))$ in the triangulated category $\mathbf{D}^b(\mathscr{A})$. It can be proven that the intersection $\mathscr{X}(\mathscr{S}) \cap \Sigma(\mathscr{Y}(\mathscr{G}))$ is an abelian category which is equivalent to $\mathbf{mod} - (\mathrm{End}_A(T))$ and that $\mathbf{D}^b(\mathscr{A})$ is equivalent to $\mathbf{D}^b(\mathbf{mod} - (\mathrm{End}_A(T)))$. This gives the flavour of possible connections between tilting theory and localization at the level of derived category.

Torsion pairs in a triangulated category $(\mathscr{T},\ \Sigma)$ are also related to *t-structures*, which are essentially couples $(\mathscr{X},\ \Sigma\mathscr{Y})$, where $(\mathscr{X},\ \mathscr{Y})$ is a torsion pair. In the recent works [FL14] and [FL15], the authors study t-structures in the context of *stable* $\infty-$*categories* (whose homotopy categories admits a canonical triangulation) and relate them to factorization systems and torsion theories within the same setting. The authors claim that this new environment seems to give insightful perspectives and connections also on constructions in the classical theory of triangulated categories which may then find benefits from such a higher categorical point of view.

The Telescope Conjecture Let us consider a *pointed* model category $\mathscr{M}$. Using the theory of *framings*, it is possible to show that the homotopy category $\mathrm{Ho}(\mathscr{M})$ of $\mathscr{M}$ is a closed $\mathrm{Ho}(\mathbf{Sset}_*)-$module (see [Hov], Chapters 4 and 5). Briefly, one constructs bifunctors

$$(-)\wedge(?)\colon \mathscr{M}\times\mathbf{Sset}_*\longrightarrow\mathscr{M},\quad \mathrm{Hom}_*(?,\ -)\colon \mathbf{Sset}_*\times\mathscr{M}^{op}\longrightarrow\mathscr{M}^{op}$$

which, despite not being Quillen bifunctors in general, still admit total left derived functor

$$(-)\wedge^L(?)\colon\ \mathrm{Ho}(\mathscr{M})\times\mathrm{Ho}(\mathbf{Sset}_*)\longrightarrow\mathrm{Ho}(\mathscr{M}),\quad \mathrm{RHom}_*(?,\ -)\colon\ \mathrm{Ho}(\mathbf{Sset}_*)\times\mathrm{Ho}(\mathscr{M})^{op}\longrightarrow\mathrm{Ho}(\mathscr{M})^{op}$$

respectively. These two functors fit into an adjunction of two variables $\mathrm{Ho}(\mathscr{M})\times\mathrm{Ho}(\mathbf{Sset}_*)\longrightarrow\mathrm{Ho}(\mathscr{M})$ which turns $\mathrm{Ho}(\mathscr{M})$ into a module over the closed symmetric monoidal category $\mathrm{Ho}(\mathbf{Sset}_*)$. In particular, one gets a couple of adjoints functor

$$\Sigma\colon\ \mathrm{Ho}(\mathscr{M})\longrightarrow\mathrm{Ho}(\mathscr{M}),\quad X\mapsto X\wedge^L S^1$$

$$\Omega\colon\ \mathrm{Ho}(\mathscr{M})\longrightarrow\mathrm{Ho}(\mathscr{M}),\quad X\mapsto \mathrm{RHom}_*(S^1,\ X)$$

where Σ is the left adjoint and S^1 is the pointed simplicial circle (given as the quotient of the standard (pointed) 1-simplex by its boundary). The functors Σ and Ω are called *suspension* and *loop functor* respectively, as, when $\mathscr{M}$ is, say, $\mathbf{CgHaus}_*$, the category of compactly generated, Hausdorff, pointed topologial spaces with the Quillen model structure, they coincide with the usual (reduced) suspension and loop space used in classical Algebraic Topology. These functors give rise to an incredible amount of structure on $\mathrm{Ho}(\mathscr{M})$. Indeed, using a coaction of the cogroup ΣA for a cofibrant object A in $\mathscr{M}$, one can define *cofiber sequences* in $\mathrm{Ho}(\mathscr{M})$ as specific diagrams

$$X\longrightarrow Y\longrightarrow Z$$

together with a right coaction of ΣX on Z (hence, in particular, with a map $Z\to\Sigma X$). These cofiber sequences are also called *left triangles*, as their collection satisfy a set of actions wich are completely analogous to the ones defining a triangulated structure on a (additive) category (see [Hov] §6.5). For example, cofiber sequences are replete, every map can be completed to a cofiber sequence and Verdier's "ochtaedral axiom" holds. However, in general, the homotopy category of a pointed model category is *not* a triangulated category with respect to the class of cofiber sequences, as the suspension functor Σ lacks to be an equivalence of categories. For example, $\mathbf{CgHaus}_*$ or $\mathbf{CW}_*$, the category of pointed CW complexes, are definitely not stable. This leads to the following

Definition 3.4. A pointed model category $\mathscr{M}$ is called a *stable model category* if the suspension functor $\Sigma\colon\ \mathrm{Ho}(\mathscr{M})\longrightarrow\mathrm{Ho}(\mathscr{M})$ is an equivalence.

The homotopy category of a stable model category is a triangulated category (in particular, it is additive) with distinguished triangles induced by cofiber sequences with the further property of being a closed Ho($\mathbf{Sset}_*$)−module. A classical example of this kind of model categories comes from topology, as the category of (Bousfield–Friedlander) spectra with the model structure given by stable weak equivalences and stable (co)fibrations is a stable model category. Stable model categories also include the category $\mathbf{Ch}(R)$ of chain complexes on the category of (left) modules over a ring R with the model structure, for example, where the weak equivalences are the quasi-isomorphisms and the cofibrations are the monomorphisms. However, if R is a commutative ring, $\mathbf{Ch}(R)$ is a (closed) symmetric monoidal model category (with respect to the so-called *projective model structure* where weak equivalences are quasi-isomorphisms and fibrations are epimorphisms) with the usual tensor product of chain complexes. As a consequence, Ho($\mathbf{Ch}(R)$), which is the derived category of R, is not only a triangulated category, but it is a closed symmetric monoidal category itself and such an added structure properly interacts with the triangulated one.

To capture and axiomatize the most fundamental properties of such structured triangulated categories, the authors of [HPS] define the notion of a *stable homotopy category*, which is essentially a triangulated category $\mathscr{C}$ with a closed symmetric monoidal structure compatible with the triangulation (in a precise sense), which also admits arbitrary coproducts of objects of $\mathscr{C}$ and is generated by a small set $\mathcal{G}$ of objects (required to satisfy a technical property called *strong dualizability*). It is also assumed that every cohomological functor on $\mathscr{C}$ is representable, which we already remarked to be automatically true if $\mathscr{C}$ is perfectly generated by a small set of objects, thanks to Brown's representabilty theorem. A stable homotopy category such that the generating set $\mathcal{G}$ is made of (strongly dualizable) *small* objects[1] is called an *algebraic* stable homotopy category.[2] The homotopy category of spectra and the derived category of a commutative ring are both examples of algebraic stable homotopy categories (in this case $\mathcal{G}$ consists actually of one single object, namely the singleton of the unit of the monoidal structure). However, in principle, not every stable (symmetric monoidal) model categories gives rise to a (weakened version of a) stable homotopy category, as further assumptions needs to be made: see [Hov], Theorem 7.2.5. Apparently, (algebraic) stable homotopy categories also arises in Algebraic Geometry as derived categories of the Grothendieck category of quasi-coherent sheaves of $\mathcal{O}_X$−modules over nice schemes, see [AJPV].

Much of the theory of localization for general triangulated categories can be reformulated and reinterpreted in the stable setting. For example, one can adapt the notions of localization and colocalization functors and show that there is a natural equivalence between localization functors L and colocalization functors C on a stable homotopy category $\mathscr{C}$ with suspension functor Σ, in which L and C correspond one to the other if and only if, for each $X \in \mathscr{C}$, the sequence

$$CX \to X \to LX \longrightarrow \Sigma CX$$

is a distinguished triangle (here the maps are induced by the natural transformations $\mathrm{Id} \longrightarrow L$ and $C \longrightarrow \mathrm{Id}$ which are part of the definition of a localization and colocalization functors). Moreover, given a localization functor L on a stable homotopy category $\mathscr{C}$, the subcategory of L−local objects is still a stable homotopy category and, if $\mathscr{C}$ is algebraic, then for every cohomological functor $H\colon \mathscr{C} \longrightarrow \mathbf{Ab}$, there is a localization functor L_H on $\mathscr{C}$ such that the class of acyclics objects for L_H is the full subcategory of X in $\mathscr{C}$ such that $H(X \wedge Y) = 0$ for all $Y \in \mathscr{C}$, where $\wedge\colon \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ is the bifunctor giving the monoidal structure on $\mathscr{C}$.

In this context, one can formulate, the so-called *telescope conjecture* which is said to hold in a stable homotopy category if every *smashing localization* of $\mathscr{C}$ is a *finite localization* (we refer to §3.3 of [HPS] for the definitions). Apparently, this conjecture is the algebraic translation of a homonym question which was asked in the theory of spectra in [Rav]. Both [HPS] (section 10) and [Hov] (see e.g. Problems 8.11, 8.12 and 8.14) gives some possible inputs and vistas in this area.

[1]In this context, an object $X \in \mathscr{C}$ is small if the covariant Hom functor $\mathscr{C}(X, -)$ preserves coproducts.

[2]In [HPS] the authors consider and use also weaker and stronger notions than the one of a (algebraic) stable homotopy category. We have decided to omit them in this context, as they do not seem to be relevant here.

References

[AdRo] J. Adámek, J. Rosický, *Locally Presentable and Accessible Categories*, Cambridge University Press, Cambridge, 1994.

[AJPV] L. Alonso, A. Jeremias, M. Perez, M. J. Vale, *The derived category of quasi-coherent sheaves and axiomatic stable homotopy*, `http://arxiv.org/abs/0706.0493`.

[BaPa] S. Bazzoni, A. Pavarin, *Recollements from partial tilting complexes*, Journal of Algebra Volume 388, 2013.

[BeRe] A. Beligiannis, I. Reiten, *Homological and Homotopical Aspects of Torsion Theories*, Memoirs of the American Mathematical Society, 2007.

[FL14] D. Fiorenza, F. Loregian, *t-structures are normal torsion theories*, `http://arxiv.org/abs/1408.7003`.

[FL15] D. Fiorenza, F. Loregian, *Hearts and towers in stable infinity-categories*, `http://arxiv.org/abs/1501.04658`.

[GaZi] P. Gabriel, M. Zisman, *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag New York 1967.

[Hov] M. Hovey, *Model Categories*, AMS, Mathematical Surveys and Monographs, Volume 63, 2007.

[HPS] M. Hovey, J. H. Palmieri, N. P. Strickland, *Axiomatic Stable Homotopy Theory*, Memoirs of the American Mathematical Society 1997.

[Kr1] H. Krause, *Localization theory for triangulated categories*, Triangulated categories, 161–235, London Math. Soc. Lecture Note Ser. 375, Cambridge Univ. Press, Cambridge, 2010.

[Kr2] H. Krause, *Derived categories, resolutions, and Brown representability*, `http://arxiv.org/abs/math/0511047`.

[Kr3] H. Krause, *Smashing subcategories and the telescope conjecture - an algebraic approach*, Invent. Math. Vol. 139, 2000.

[Nee96] A. Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc. 9, 1996.

[Nee01] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies 148, Princeton University Press, 2001.

[Nic] P. Nicolás Saragoza, *On torsion torsionfree triples*, `http://arxiv.org/abs/0801.0507`.

[Rav] D. C. Ravanel, *Localization with Respect to Certain Periodic Homology Theories*, American Journal of Mathematics, Vol. 106, No. 2, 1984.