Mathematics 222a Final Exam December 16, 2001

Instructions: Print your name, your student number, your course section and your instructor's name on the Scantron answer sheet, and sign the Scantron sheet. Use a PENCIL to code your student number and course section on the Scantron answer sheet. For each of questions A1–A30 below, circle your answer on the question sheet and use a PENCIL to mark your answer on the Scantron answer sheet.

A1. [*2 marks*] In how many ways can the letters in the word MONOTONOUS be arranged?

A: $10!$	B: $\binom{10}{4}$	C: $10!/(4!2!)$	D: $10!/4!$	E: $10!/4$

A2. [*2 marks*] In how many ways can the letters in the word MONOTONOUS be arranged if we require that every N must appear before any O appears?

A: $\binom{10}{6}4!$	B: $6!$	C: $10!/(4!2!)$	D: $4!$	E: None of A, B, C, D

A3. [*2 marks*] In how many ways can the letters in the word MONOTONOUS be arranged if we require that no two O's be adjacent?

A: $10!/4!$	B: $\binom{7}{4}$	C: $10!/(4!3!)$	D: $360\binom{7}{3}$	E: None of A, B, C, D

A4. [*2 marks*] The number of non-negative integer solutions of $x_1 + x_2 + x_3 = 14$ is:

A: $16!$	B: $\binom{16}{3}$	C: $\binom{16}{2}$	D: $\binom{17}{3}$	E: $(14)(13)(12)$

A5. [*2 marks*] The sum of the coefficients in the expansion of $(4x - 7y + 3z)^{101}$ is:

A: -1	B: 1	C: 3^{101}	D: 0	E: None of A, B, C, D

A6. [*2 marks*] If $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then the number of subsets of A which contain at least one of the elements 1 or 2, but neither of the elements 4 and 5 is:

A: $10!/(8!2!)$	B: $10!/(2!2!)$	C: 2^6	D: $2^8 - 2^6$	E: None of A, B, C, D

A7. [*2 marks*] The value of $\sum_{i=0}^{195} \binom{195}{i}(-1)^i 2^i$ is:

A: -1	B: 1	C: 0	D: $(-3)^{195}$	E: None of A, B, C, D

A8. [*2 marks*] Exactly which of the following are true for all subsets A, B and C of a given set S?

(i) $(A - B) \cup C = (C - B) \cup A$.
(ii) $(A - B) \cap C = (C - B) \cap A$.
(iii) $A \cap (A^c \cup C^c) = A - C$.
(iv) $A \subseteq B$ if and only if $A^c \cup B = S$.

A: (ii), (iii)	B: (ii), (iv)	C: (iii), (iv)	D: (i), (iii), (iv)	E: (ii), (iii), (iv)

A9. [*2 marks*] Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The number of functions $f : A \to B$ is:

A: 4^3	B: 12^{12}	C: 4×3	D: 2^{12}	E: 3^4

A10. [*2 marks*] If $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$, then $|A \times B|$ is equal to:

A: 4^3	B: $4!$	C: 12	D: 2^{12}	E: 3^4

A11. [*2 marks*] Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The number of relations from A to B is:

A: 4^3	B: $3!$	C: 2^{12}	D: 3^4	E: 4×3

A12. [*2 marks*] Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The number of surjective functions $f : B \to A$ (note direction) is:

A: 0	B: 60	C: 24	D: 36	E: None of A, B, C, D

A13. [*2 marks*] Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The number of left inverses of an injective function $f : A \to B$ is:

A: 0	B: 3	C: 2	D: 1	E: None of A, B, C, D

A14. [*2 marks*] How many symmetric relations on the set $A = \{1, 2, 3\}$ contain $(1, 1)$?

A: 16	B: $2^2 3^3$	C: 2^6	D: $2^3 - 2$	E: 32

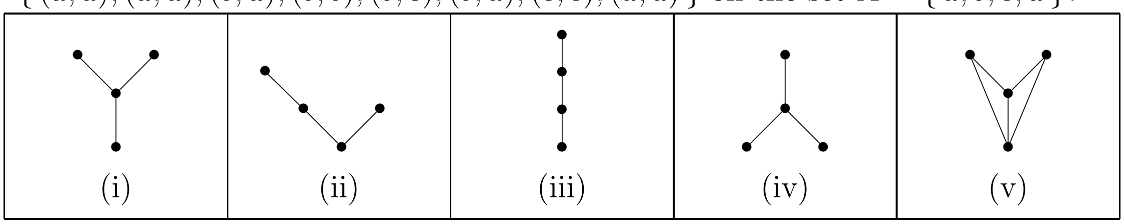
A15. [*2 marks*] The number of relations on a three element set that are both symmetric and antisymmetric is:

A: 2^4	B: 7	C: 2^6	D: 2^3	E: 3^2

A16. [*2 marks*] The number of relations on a three element set that are reflexive and symmetric but not transitive is:

A: 2^3	B: 2^6	C: 0	D: 5	E: 3

A17. [*2 marks*] Which of the following diagrams is the Hasse diagram of the partial order relation
$R = \{(a, a), (a, d), (b, a), (b, b), (b, c), (b, d), (c, c), (d, d)\}$ on the set $A = \{a, b, c, d\}$?



A: (i)	B: (ii)	C: (iii)	D: (iv)	E: (v)

A18. [*2 marks*] Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be partially ordered by the division relation (that is, for $a, b \in A$, we say that $a \leq b$ if a is a divisor of b). How many maximal elements are there for this partial order relation?

A: 1	B: 2	C: 3	D: 4	E: 5

A19. [*2 marks*] Exactly which of the following statements must be true in any group $(G, *)$?

(i) For any given $a, x, y \in G$, if $a * x = a * y$, then $x = y$;
(ii) For any given a and b in G, the equation $a * x = b$ always has a solution in G;
(iii) If G is finite, then there must an element of order 2.

A: (i)	B: (ii)	C: (iii)	D: (i), (ii)	E: (i), (iii)

A20. [*2 marks*] Exactly which of the following must be true for all subgroups H and K of a group G?

(i) $H \cup K$ is a subgroup of G;
(ii) $H \cap K$ is a subgroup of G;
(iii) $H \triangle K$ is a subgroup of G.

A: (i)	B: (ii)	C: (iii)	D: (ii), (iii)	E: None of A, B, C, D

A21. [*2 marks*] If G is a group of order 20 and H is a subgroup of G of order 5, then the number of left cosets of H in G is:

A: 4	B: 5	C: 2	D: 1	E: None of A, B, C, D

In questions A22-A24, the group $G = \{ 1, a, b, c, d, f, g, h \}$ has its binary operation given by the following table:

	1	a	b	c	d	f	g	h
1	1	a	b	c	d	f	g	h
a	a	g	h	b	f	1	c	d
b	b	h	1	f	g	c	d	a
c	c	b	f	d	a	g	h	1
d	d	f	g	a	b	h	1	c
f	f	1	c	g	h	d	a	b
g	g	c	d	h	1	a	b	f
h	h	d	a	1	c	b	f	g

A22. [*2 marks*] In the group G above, what is ab^{-1}?

A: a	B: b	C: f	D: g	E: h

A23. [*2 marks*] Exactly which of the following are subgroups of the group G above?

(i) $\{ 1, b \}$
(ii) $\{ 1, a, b, h \}$
(iii) $\{ 1, b, d, g \}$

A: (i)	B: (ii)	C: (iii)	D: (i), (ii)	E: (i), (iii)

A24. [*2 marks*] Exactly which of the following statements are true of the group G above?

(i) G has exactly 2 elements of order 2;
(ii) G is commutative (abelian);
(iii) G has a subgroup of order 6.

A: (i)	B: (ii)	C: (iii)	D: (i), (iii)	E: None of A, B, C, D

A25. [*2 marks*] Exactly which of the following sets with operations are groups?

(i) The two element set of real numbers $\{ 0, 1 \}$ with the usual multiplication of real numbers;
(ii) The set of all ordered pairs (x, y) of real numbers, with binary operation $\star$ defined by $(x_1, y_1) \star (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$;
(iii) The set of all functions from $A = \{ 1, 2, 3 \}$ to $A = \{ 1, 2, 3 \}$, with binary operation composition of functions.

A: (i), (ii)	B: (i), (iii)	C: (ii), (iii)	D: (ii)	E: None of A, B, C, D

A26. [*2 marks*] Exactly which of the following are monoids?

(i) The set of real numbers under multiplication.
(ii) The set of all even integers under multiplication.
(iii) The set of all integers under addition.

A: (i)	B: (i), (iii)	C: (iii)	D: (ii)	E: None of A, B, C, D

A27. [*2 marks*] Let $*$ denote the binary operation defined on $\mathbb{Z}$ by $x * y = x - y$ for all $x, y \in \mathbb{Z}$. Exactly which of the following properties does $*$ have?

(i) It is commutative.
(ii) It is associative.
(iii) It has an identity element.

A: (i), (ii)	B: (i), (iii)	C: (ii), (iii)	D: All of them	E: None of A, B, C, D

A28. [*2 marks*] Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3\}$. How many surjective functions f are there from A to B with the property that $f(1) = 1$ and $f(2) = 2$?

A: 150	B: 3!	C: 3	D: 19	E: 27

A29. [*2 marks*] Let $A = \{1, 2, 3\}$. The number of binary operations on A that have 2 as an identity element is:

A: 2^9	B: 3^9	C: 1	D: 3^4	E: 3^3

A30. [*2 marks*] Let A be a set of size 5. Exactly which of the following is the number of ordered pairs in some equivalence relation on A?

(i) 13 (ii) 9 (iii) 10

A: (i)	B: (ii)	C: (i), (ii)	D: (i), (iii)	E: All of them

PART B

B1. [*3 marks*] Let G_1 and G_2 be groups, and let $G_1 \times G_2$ denote the direct product group. Prove that if G_1 and G_2 are abelian, then $G_1 \times G_2$ is abelian. For simplicity, use $*$ to denote the binary operation in each of G_1, G_2 and in $G_1 \times G_2$, so that for (a, b), $(c, d) \in G_1 \times G_2$, we have

$$(a, b) * (c, d) = (a * c, b * d).$$

(You do not need to prove that $G_1 \times G_2$ is a group.)

B2. [*4 marks*] Let $\mathbb{R}$ denote the set of real numbers with the usual order relation. You may assume that the relation

$$P = \{\ (\,(x, y), (r, s)\,)\ \big|\ \text{either } x < r \text{ or else both } x = r \text{ and } y \leq s\ \}$$

is a partial order relation on $\mathbb{R} \times \mathbb{R}$. Prove that P is a total order relation (that is, for each (x_1, y_1), $(x_2, y_2) \in \mathbb{R} \times \mathbb{R}$, $(\,(x_1, y_1), (x_2, y_2)\,) \in P$ or $(\,(x_2, y_2), (x_1, y_1)\,) \in P$).

B3. [*4 marks*] Prove that for all sets A and B, $A \cup (A \vartriangle B) = B \cup (A \vartriangle B)$.

B4. [*4 marks*] If G is a group of size 15 and H is a subgroup of G of size at least 6, determine the size of H. Provide reasons for your answer.

B5. [*4 marks*] Let G, H and K be groups, and let $f : G \to H$ and $g : H \to K$ be homomorphisms. Prove that $g \circ f$ is a homomorphism (for simplicity, use $*$ to denote the binary operation in each group).

B6. Define a binary operation $*$ on $\mathbb{Q}$, the set of rational numbers, by the rule $a * b = 3ab$, where we are using the usual multiplication on $\mathbb{Q}$ to calculate $3ab$.

(a) [*2 marks*] Prove that $*$ is commutative.
(b) [*2 marks*] Prove that $*$ is associative.
(c) [*3 marks*] Prove that $*$ has an identity.

B7. [*4 marks*] Let $a_0 = 2$, and for each integer $n \geq 1$, let $a_n = \sqrt{a_{n-1} + 1}$. Use mathematical induction to prove that $a_n \geq a_{n+1}$ for all integers $n \geq 0$.

B8. [*2 marks*] Let $A = \{1, 2, 3, 4\}$, let $B = \{a, b, c, d\}$ and let $f : A \to B$ be the function

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & c \end{pmatrix}.$$

Write down the partition of A that is associated with the equivalence relation $\mathrm{Ker}(f)$. In other words, write down $A/\mathrm{Ker}(f)$.

B9. Consider again the group $G = \{1, a, b, c, d, f, g, h\}$ with binary operation as given by the following table:

	1	a	b	c	d	f	g	h
1	1	a	b	c	d	f	g	h
a	a	g	h	b	f	1	c	d
b	b	h	1	f	g	c	d	a
c	c	b	f	d	a	g	h	1
d	d	f	g	a	b	h	1	c
f	f	1	c	g	h	d	a	b
g	g	c	d	h	1	a	b	f
h	h	d	a	1	c	b	f	g

(a) [*3 marks*] Establish whether or not the subset $H = \{1, a, g, b\}$ is a subgroup of G. Explain your answer.

(b) [*3 marks*] Compute $\langle g \rangle$, the subgroup generated by g. Show your work.

(c) [*2 marks*] You may assume that $H = \{1, b\}$ is a subgroup of G. Find the left coset cH.