Instructions: Print your name, your student number, your course section and your instructor's name on the Scantron answer sheet, and sign the Scantron sheet. Use a PENCIL to code your student number and course section on the Scantron answer sheet. For each of questions A1–A30 below, circle your answer on the question sheet and use a PENCIL to mark your answer on the Scantron answer sheet.

2 marks A1. In how many ways can the letters in the word MONOTONOUS be arranged?

A: $10!$	B: $\binom{10}{4}$	C: $10!/(4!2!)$	D: $10!/4!$	E: $10!/4$

Solution: Since there is 1 letter M, 4 letter O's, 2 letter N's, 1 letter T, 1 letter U, and 1 letter S, for a total of 10 letters, there are $10!/(4!2!1!1!1!1!) = 10!/(4!2!)$ ways to arrange the letters.
The answer is C.

2 marks A2. In how many ways can the letters in the word MONOTONOUS be arranged if we require that every N must appear before any O appears?

A: $\binom{10}{6}4!$	B: $6!$	C: $10!/(4!2!)$	D: $4!$	E: None of A, B, C, D

Solution: There are $\binom{10}{6}$ ways to choose six spots for N,N,O,O,O,O, placing them in that order, which leaves $10 - 6 = 4$ letters to arrange. Since the remaining 4 letters are all different, there are $4!$ ways to arrange them, so altogether, we have $\binom{10}{6}4!$ such arrangements.
The answer is A.

2 marks A3. In how many ways can the letters in the word MONOTONOUS be arranged if we require that no two O's be adjacent?

A: $10!/4!$	B: $\binom{7}{4}$	C: $10!/(4!3!)$	D: $360\binom{7}{3}$	E: None of A, B, C, D

Solution: Position the four O's in a row, leaving gaps between them. Now let x_1 denote the number of letters to place to the left of the first O, x_2 the number of letters to place between the first and the second O's, x_3 the number between the second and the third O's, and x_4 be the number of letters to place to the right of the fourth and last O. Then there are as many ways to allocate places for the $10 - 4 = 6$ non-O letters as there are solutions to the equation $x_1 + x_2 + x_3 + x_4 = 6$, subject to the constraints $x_1 \geq 0$, $x_2 \geq 1$, $x_3 \geq 1$, and $x + 4 \geq 0$. If we allocate in advance one letter place between the first and second O's, and one letter place between the second and third O's, then we can recode x_2 and x_3 to count the number of letters over and above 1 to place in the respective places. The number of such allocations of spaces is thus the number of solutions to the equation $x_1 + x_2 + x_3 + x_4 = 4$, subject to the constraints $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, and $x + 4 \geq 0$, which is $(4 + 3)!/(4!3!)$, or $\binom{7}{3}$. For each such allocation, we may place the 6 remaining letters in $6!/2!$ ways. Altogether, there are $\binom{7}{3}6!/2!$ ways to arrange the letters of the word MONOTONOUS so that no two $O's$ are adjacent.
The answer is D.

2 marks A4. The number of non-negative integer solutions of $x_1 + x_2 + x_3 = 14$ is:

A: $16!$	B: $\binom{16}{3}$	C: $\binom{16}{2}$	D: $\binom{17}{3}$	E: $(14)(13)(12)$

Solution: We are looking for solutions (x_1, x_2, x_3) to $x_1 + x_2 + x_3 = 14$ subject to the constraint that x_1, x_2, x_3 are integers satisfying $x_1 \geq 0$, $x_2 \geq 0$, and $x_3 \geq 0$. This is $(14 + 2)!/(14!2!)$, which is equal to $\binom{16}{2}$.
The answer is C.

2 marks A5. The sum of the coefficients in the expansion of $(4x - 7y + 3z)^{101}$ is:

A: -1	B: 1	C: 3^{101}	D: 0	E: None of A, B, C, D

Solution: The multinomial expansion of $(4x - 7y + 3z)^{101}$ is

$$(4x - 7y + 3z)^{101} = \sum_{\substack{n_1+n_2+n_3=101 \\ n_1\geq 0, n_2\geq 0, n_3\geq 0}} \binom{101}{n_1, n_2, n_3} (4x)^n_1 (-7y)^n_2 (3z)^n_3$$

$$= \sum_{\substack{n_1+n_2+n_3=101 \\ n_1\geq 0, n_2\geq 0, n_3\geq 0}} (-1)^n_2 \binom{101}{n_1, n_2, n_3} 4^n_1 7^n_2 3^n_3 x^n_1 y^n_2 z^n_3$$

and so the sum of the coefficients is

$$\sum_{\substack{n_1+n_2+n_3=101\\ n_1\geq 0, n_2\geq 0, n_3\geq 0}} (-1)_2^n \binom{101}{n_1, n_2, n_3} 4_1^n 7_2^n 3_3^n.$$

This is the result of setting $x = y = z = 1$ in the multinomial expansion, so the sum of the coefficients is $(4-7+3)^{101} = 0$.
The answer is D.

2 marks A6. If $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then the number of subsets of A which contain at least one of the elements 1 or 2, but neither of the elements 4 and 5 is:

A: $10!/(8!2!)$	B: $10!/(2!2!)$	C: 2^6	D: $2^8 - 2^6$	E: None of A, B, C, D

Solution: To form such subsets, we set 4 and 5 aside, and then make any choice whatsoever from the collection $\{1, 2, 3, 6, 7, 8, 9, 10\}$ subject to the requirement that we choose at least one of 1 or 2. The number of subsets of an eight element set is 2^8. The number that contain neither 1 nor 2 is 2^6, so the number that contain at least one of these elements is $2^8 - 2^6$.
The answer is D.

2 marks A7. The value of $\sum_{i=0}^{195} \binom{195}{i}(-1)^i 2^i$ is:

A: -1	B: 1	C: 0	D: $(-3)^{195}$	E: None of A, B, C, D

Solution: $(1-2)^{195} = \sum_{i=0}^{195} \binom{195}{i}(-2)^i 1^{195-i} = \sum_{i=0}^{1} 95\binom{195}{i}(-1)^i 2^i$, so the value is $(-1)^{195} = -1$.
The answer is A.

2 marks A8. Exactly which of the following are true for all subsets A, B and C of a given set S?

(i) $(A - B) \cup C = (C - B) \cup A$.
(ii) $(A - B) \cap C = (C - B) \cap A$.
(iii) $A \cap (A^c \cup C^c) = A - C$.
(iv) $A \subseteq B$ if and only if $A^c \cup B = S$.

A: (ii), (iii)	B: (ii), (iv)	C: (iii), (iv)	D: (i), (iii), (iv)	E: (ii), (iii), (iv)

Solution:

i) False. $(A-B)\cup C = (A\cap B^c)\cup C = (A\cup C)\cap(B^c\cup C)$, $(C-B)\cup A = (C\cap B^c)\cup A = (A\cup C)\cap(B^c\cup A)$. For a counterexample, we must have $B^c \cup C \neq B^c \cup A$. Try extremes, say $C = \emptyset$. In this case, the equation reduces to $A - B = A$, so take $B = A \neq \emptyset$, say $A = \{1\} = B$. Then $(A - B) \cup C = \emptyset$, while $(C - B) \cup A = \{1\}$.
ii) True, since $(A - B) \cap C = A \cap B^c \cap C = A \cap (C \cap B^c) = A \cap (C - B)$.
iii) True, since $A \cap (A^c \cup C^c) = (A \cap A^c) \cup (A \cap C^c) = A \cap C^c = A - C$.
iv) True, since $A \subseteq B \iff A \cap B = A = A \cap S = A \cap (B \cup B^c) = (A \cap B) \cup (A \cap B^c) \iff A \cap B^c \subseteq A \cap B \iff A \cap B^c = (A \cap B) \cap (A \cap B^c) = A \cap B \cap B^c = \emptyset \iff A^c \cup B = S$.

Since (i) is false, while (ii), (iii) and (iv) are true, the answer is E.

2 marks A9. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The number of functions $f : A \to B$ is:

A: 4^3	B: 12^{12}	C: 4×3	D: 2^{12}	E: 3^4

Solution: To construct a function from A to B, we simply assign to each blank in the prototype $\begin{pmatrix} 1 & 2 & 3 \\ \_ & \_ & \_ \end{pmatrix}$ an arbitrary choice of an element of B, so there are 4 choices for each blank. Thus there are $(4)(4)(4) = 4^3$ functions from A to B.
The answer is A.

2 marks A10. If $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$, then $|A \times B|$ is equal to:

A: 4^3	B: $4!$	C: 12	D: 2^{12}	E: 3^4

Solution: $|A \times B| = |A||B| = (4)(3) = 12$.
The answer is C.

2 marks A11. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The number of relations from A to B is:

A: 4^3	B: $3!$	C: 2^{12}	D: 3^4	E: 4×3

Solution: A relation from A to B is simply a subset of $A \times B$. Since $A \times B$ has size $|A||B| = (3)(4) = 12$, there are 2^{12} subsets of $A \times B$, hence there are 2^{12} relations from A to B.

The answer is C.

2 marks A12. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The number of surjective functions $f : B \to A$ (note direction) is:

A: 0	B: 60	C: 24	D: 36	E: None of A, B, C, D

Solution: This is the value of $S(4, 3)$. Since the set sizes differ by 1, we may determine this value easily by counting the number of ways of partitioning the four element set A into 3 cells, then assign in a one-to-one fashion the 3 cells to the three elements of B. Any partition of A with 3 cells has exactly one cell of size 2 and two of size 1, so the number of such partitions of A is the number of choices of 2 from 4; namely $\binom{4}{2}$. Since there are 3! ways to match the 3 cells with the 3 elements of B, there are $\binom{4}{2}\, 3! = (6)(6) = 36$ surjective functions from A to B.

Alternatively, we could use the recursive method. We have $S(4, 3) = 3[S(3, 3) + S(3, 2)] = 3(3!) + 3S(3, 2) = 18 + (3)(2)[S(2, 2) + S(2, 1)] = 18 + 6(2! + 1) = 18 + 18 = 36$.

Finally, one could use the summation expression $S(4, 3) = (-1)^3 \sum_{i=1}^{3} (-1)^i \binom{3}{i} i^4 = -[-3+3(2^4)-3^4] = 3 - 48 + 81 = 84 - 48 = 36$.

The answer is D.

2 marks A13. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4\}$. The number of left inverses of an injective function $f : A \to B$ is:

A: 0	B: 3	C: 2	D: 1	E: None of A, B, C, D

Solution: To construct a left inverse of an injective function $f : A \to B$, we construct the relation f^{-1}, which can be considered as a bijective function from $\mathrm{Im}(f)$ to A, and then extend this to a function from B to A. Since the injectivity of f implies that $|\mathrm{Im}(f)| = |A|$, we see that $B - \mathrm{Im}(f)$ consists of a single element, so the extension of f^{-1} to a function from B to A is accomplished by determining which element of A shall be assigned to the element of $B - \mathrm{Im}(f)$. Since there are 3 elements in A, there are 3 choices, and so there are 3 left inverses of an injective function from A to B.

The answer is B.

2 marks A14. How many symmetric relations on the set $A = \{1, 2, 3\}$ contain $(1, 1)$?

A: 16	B: $2^2 3^3$	C: 2^6	D: $2^3 - 2$	E: 32

Solution: A symmetric relation R on A is made up by choosing to include or exclude each of the following subsets of $A \times A$ in R: $\{(1, 1)\}$, $\{(2, 2)\}$, $\{(3, 3)\}$, $\{(1, 2), (2, 1)\}$, $\{(1, 3), (3, 1)\}$, and $\{(2, 3), (3, 2)\}$. Since we are told that we must choose to include $\{(1, 1)\}$, we must make our choices from the remaining 5 subsets. Each subset presents us with 2 choices, take it or leave it, so there are 2^5 such relations.

The answer is E.

2 marks A15. The number of relations on a three element set that are both symmetric and antisymmetric is:

A: 2^4	B: 7	C: 2^6	D: 2^3	E: 3^2

Solution: It is a result in the text that a relation is both symmetric and antisymmetric if and only if it is a subset of the identity relation. Since the identity relation on a three element set contains 3 ordered pairs, there are 2^3 subsets of the identity relation on a three element set, so there are 2^3 relations on a three element set that are both symmetric and antisymmetric.

The answer is D.

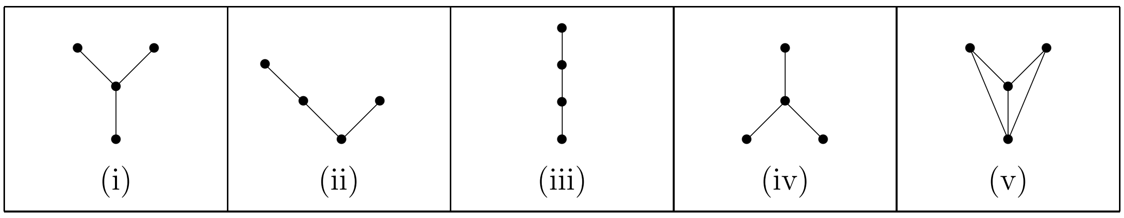
2 marks A16. The number of relations on a three element set that are reflexive and symmetric but not transitive is:

A: 2^3	B: 2^6	C: 0	D: 5	E: 3

Solution: We determine the number of relations on a three element set that are reflexive and symmetric, and the number of relations on a three element set that are reflexive, symmetric, and transitive, and calculate the difference. There are 2^3 relations on a three element set that are both reflexive and symmetric. Since a relation is reflexive, symmetric, and transitive if and only if it is an equivalence relation, and since there is a one-to-one matchup between equivalence relations of a set and partitions of the set, we shall determine B_3, the third Bell number, which gives the number of partitions of a three element set. $B_3 = \sum_{i=0}^{2} \binom{2}{i} B_i = B_0 + 2\,B_1 + B_2$, where $B_0 = B_1 = 1$ and $B_2 = 2$. Thus $B_3 = 1 + 2 + 2 = 5$. We could of course determine B_3 directly, since a partition of a three element set could have 3 singleton cells, or 1 cell of size 2 and 2 cells of size 1, or 1 cell of size 3, so there are $1 + \binom{3}{2} + 1 = 1 + 3 + 1 = 5$ partitions of a three element set. Thus there are 5 equivalence relations on a three element set, and so there are $2^3 - 5 = 3$ relations on a three element set that are reflexive and symmetric but not transitive.

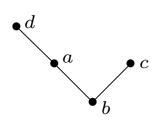
The answer is E.

2 marks A17. Which of the following diagrams is the Hasse diagram of the partial order relation $R = \{ (a,a), (a,d), (b,a), (b,b), (b,c), (b,d), (c,c), (d,d) \}$ on the set $A = \{ a, b, c, d \}$?



A: (i)	B: (ii)	C: (iii)	D: (iv)	E: (v)

Solution: Since $R - 1_A = \{ (a,d), (b,a), (b,c), (b,d) \}$, we have $(R - 1_A) \circ (R - 1_A) = \{ (b,d) \}$ and so $R^o = \{ (a,d), (b,a), (b,c), (b,d) \} - \{ (b,d) \} = \{ (a,d), (b,a), (b,c) \}$. Thus the Hasse diagram for R is

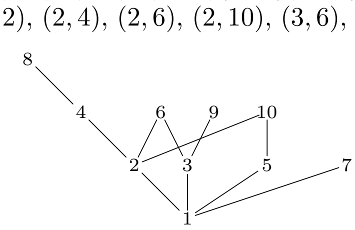


The answer is B.

2 marks A18. Let $A = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}$ be partially ordered by the division relation (that is, for $a, b \in A$, we say that $a \leq b$ if a is a divisor of b). How many maximal elements are there for this partial order relation?

A: 1	B: 2	C: 3	D: 4	E: 5

Solution: This partial order relation is $R = \{ (1,n) \mid n = 1..10 \} \cup \{ (2,2), (2,4), (2,6), (2,8), (2,10) \} \cup \{ (3,3), (3,6), (3,9) \} \cup \{ (4,4), (4,8) \} \cup \{ (5,5), (5,10) \} \cup \{ (6,6), (7,7), (8,8), (9,9), (10,10) \}$, so $R - 1_A = \{ (1,n) \mid n = 2..10 \} \cup \{ (2,4), (2,6), (2,8), (2,10) \} \cup \{ (3,6), (3,9) \} \cup \{ (4,8) \} \cup \{ (5,10) \}$ and thus $(R - 1_A) \circ (R - 1_A) = \{ (1,4), (1,6), (1,8), (1,10) \} \cup \{ (2,8) \}$. The Hasse diagram is the graph of $R - 1_A - (R - 1_A) \circ (R - 1_A) = \{ (1,2), (2,4), (2,6), (2,10), (3,6), (3,9), (4,8), (5,10) \}$.



Alternatively, observe that an element of A is maximal if it doesn't divide any other elements of A. The elements 1, 2, 3, 4, and 5 divide the elements 2, 4, 6, 8, and 10, respectively. Since any multiple greater than 1 of any of 6, 7, 8, 9 and 10 is greater than 10, none of these can divide any element of A other than themselves.

The maximal elements of this partial order are $6, 7, 8, 9, 10$.

The answer is E.

2 marks A19. Exactly which of the following statements must be true in any group $(G, *)$?

(i) For any given $a, x, y \in G$, if $a * x = a * y$, then $x = y$;
(ii) For any given a and b in G, the equation $a * x = b$ always has a solution in G;
(iii) If G is finite, then there must an element of order 2.

A: (i)	B: (ii)	C: (iii)	D: (i), (ii)	E: (i), (iii)

Solution:

(i) True, this is the cancellation law, which is valid in any group.
(ii) True, the solution is $x = a^{-1} * b$.
(iii) False. In particular, it is not true for a cyclic group of order 3, since by Lagrange's theorem, the order of any element must divide the order of the group.

Since (i) and (ii) are true while (iii) is false, the answer is D.

2 marks A20. Exactly which of the following must be true for all subgroups H and K of a group G?

(i) $H \cup K$ is a subgroup of G;
(ii) $H \cap K$ is a subgroup of G;
(iii) $H \bigtriangleup K$ is a subgroup of G.

A: (i)	B: (ii)	C: (iii)	D: (ii), (iii)	E: None of A, B, C, D

Solution:
(i) False. The simplest example would be a group of order 4 with every non-identity element of order 2, for example the direct product of a cyclic group of order 2 with itself. Such a group has 3 elements of order 2, and each element of order 2, together with the identity element, forms a subgroup of size 2. Consider the union of any two of these 2 element subgroups. The result is a subset of size 3, which can't be a subgroup of a group of size 4.
(ii) True. This was an exercise in the text.
(iii) False. Since H and K are both subgroups of G, they both contain the identity element of G, whence $H \bigtriangleup K$ does not contain the identity element of G. Since every subgroup of G must contain the identity element of G, it follows that $H \bigtriangleup K$ is never a subgroup of G.
Since only (ii) is true, the answer is B.

2 marks A21. If G is a group of order 20 and H is a subgroup of G of order 5, then the number of left cosets of H in G is:

A: 4	B: 5	C: 2	D: 1	E: None of A, B, C, D

Solution: This is the index $[G : H]$ of H in G. From the formula $|G| = [G : H]\,|H|$, we find that $20 = 5\,[G : H]$, so $[G : H] = 4$.
The answer is A.

In questions A22-A24, the group $G = \{\,1, a, b, c, d, f, g, h\,\}$ has its binary operation given by the following table:

	1	a	b	c	d	f	g	h
1	1	a	b	c	d	f	g	h
a	a	g	h	b	f	1	c	d
b	b	h	1	f	g	c	d	a
c	c	b	f	d	a	g	h	1
d	d	f	g	a	b	h	1	c
f	f	1	c	g	h	d	a	b
g	g	c	d	h	1	a	b	f
h	h	d	a	1	c	b	f	g

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2 marks A22. In the group G above, what is ab^{-1}?

A: a	B: b	C: f	D: g	E: h

Solution: From the table, we determine that the element that we should multiply b by to obtain 1 is b itself, so $b^{-1} = b$. Then from the row identified by a and the column identified by $b^{-1} = b$, we determine that $ab^{-1} = ab = h$.
The answer is E.

2 marks A23. Exactly which of the following are subgroups of the group G above?

(i) $\{\,1, b\,\}$
(ii) $\{\,1, a, b, h\,\}$
(iii) $\{\,1, b, d, g\,\}$

A: (i)	B: (ii)	C: (iii)	D: (i), (ii)	E: (i), (iii)

Solution:
(i) Since $b^2 = 1$, $< b >= \{\,1, b\,\}$, so (i) is a subgroup of G; namely the cyclic subgroup generated by b.
(ii) Since $h^2 = g \notin \{\,1, a, b, h\,\}$, (ii) is not closed under the binary operation and so is not a subgroup of G.
(iii) Since $d^2 = b$, $d^3 = db = g$, and $d^4 = dd^3 = dg = 1$, (iii) is the cyclic subgroup generated by d (and so also generated by $d^{-1} = g$).
Since (i) and (iii) are subgroups of G, but (ii) is not a subgroup of G, the answer is E.

2 marks A24. Exactly which of the following statements are true of the group G above?

(i) G has exactly 2 elements of order 2;
(ii) G is commutative (abelian);
(iii) G has a subgroup of order 6.

A: (i)	B: (ii)	C: (iii)	D: (i), (iii)	E: None of A, B, C, D

Solution:
(i) False, since an element x has order 2 if and only if x is not the identity element, but x^2 is the identity element. Thus the only element of order 2 is b.
(ii) True. The operation table is symmetric about the main diagonal, so the operation is commutative. Thus G is a commutative, or abelian, group.
(iii) False. Since the order of a subgroup of G must divide the order of G, which is 8, there can be no subgroups of G of size 6.
Since only (ii) is true, the answer is B.

2 marks A25. Exactly which of the following sets with operations are groups?

(i) The two element set of real numbers $\{0, 1\}$ with the usual multiplication of real numbers;
(ii) The set of all ordered pairs (x, y) of real numbers, with binary operation $\star$ defined by $(x_1, y_1) \star (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$;
(iii) The set of all functions from $A = \{1, 2, 3\}$ to $A = \{1, 2, 3\}$, with binary operation composition of functions.

A: (i), (ii)	B: (i), (iii)	C: (ii), (iii)	D: (ii)	E: None of A, B, C, D

Solution:
(i) This is a monoid with identity 1, but 0 is not invertible. Thus (i) is not a group.
(ii) This is the direct product group of the group of real numbers under addition with itself.
(iii) This is the monoid $F(A)$ of all functions from A to A, whose group of units is S_A, the symmtric group on A; namely the set of all bijective functions from A to A. Since there are functions from A to A that are not bijective, it follows that $F(A)$ is not a group.
Since only (ii) is a group, the answer is D.

2 marks A26. Exactly which of the following are monoids?

(i) The set of real numbers under multiplication.
(ii) The set of all even integers under multiplication.
(iii) The set of all integers under addition.

A: (i)	B: (i), (iii)	C: (iii)	D: (ii)	E: None of A, B, C, D

Solution:
(i) Multiplication is an associative binary operation on the set of real numbers, and 1 is the identity for this operation, so the real numbers under multiplication is a monoid.
(ii) The set of all even integers under is closed under multiplication, so multiplication of integers does determine an associative binary operation on the set of even integers. Since the identity element for multiplication in the set of all integers is 1, but 1 is not even, it follows that there is no identity element for the binary operation of multiplication on the set of even integers. Thus the set of even integers does not form a monoid under multiplication of integers.
(iii) The set of all integers under addition is a group (even a cyclic group), hence a monoid.
Since (ii) is not a monoid, while (i) and (iii) are monoids, the answer is B.

2 marks A27. Let $*$ denote the binary operation defined on $\mathbb{Z}$ by $x * y = x - y$ for all $x, y \in \mathbb{Z}$. Exactly which of the following properties does $*$ have?

(i) It is commutative.
(ii) It is associative.
(iii) It has an identity element.

A: (i), (ii)	B: (i), (iii)	C: (ii), (iii)	D: All of them	E: None of A, B, C, D

Solution: It is not commutative, since $0 * 1 = 0 - 1 = -1$, while $1 * 0 = 1 - 0 = 1$.
It is not associative, since $1 * (1 * 1) = 1 - (1 - 1) = 1$, while $(1 * 1) * 1 = (1 - 1) - 1 = -1$.

It does not have an identity element. For if $e \in \mathbb{N}$ satisfies $x * e = x$ for every $x \in \mathbb{N}$, then we must have $x - e = x$, so $e = 0$. Indeed, we observe that for every $x \in \mathbb{Z}$, $x * 0 = x - 0 = x$, but $0 * x = 0 - x = -x$, and $-x = x$ only if $x = 0$. Thus $*$ does not have an identity element.

The answer is E.

2 marks A28. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3\}$. How many surjective functions f are there from A to B with the property that $f(1) = 1$ and $f(2) = 2$?

A: 150	B: 3!	C: 3	D: 19	E: 27

Solution: This will be the total number of functions from A to B that send 1 to 1 and 2 to 2 minus the number of functions from A to B that send 1 to 1 and 2 to 2 but do not send anything at all to 3. This is $3^3 - 2^3 = 19$.

The answer is D.

2 marks A29. Let $A = \{1, 2, 3\}$. The number of binary operations on A that have 2 as an identity element is:

A: 2^9	B: 3^9	C: 1	D: 3^4	E: 3^3

Solution: In the operation table, the five entries for operations involving 2 are pre-determined, so there are $3^2 - 5 = 4$ entries to be specified. Each of these can be any of 1, 2 or 3, so there are 3^4 ways to complete the operation table, so 3^4 such binary operations.

The answer is D.

2 marks A30. Let A be a set of size 5. Exactly which of the following is the number of ordered pairs in some equivalence relation on A?

(i) 13 (ii) 9 (iii) 10

A: (i)	B: (ii)	C: (i), (ii)	D: (i), (iii)	E: All of them

Solution: We can write 5 as a sum of positive integers in the following ways: 5, $1+4$, $1+2+2$, $1+1+3$, $1+1+1+2$, $1+1+1+1+1$, and $2+3$, and for each of these, it is possible to partition a set of 5 elements into as many cells as there are summands, with each summand giving the size of a cell, and so for each of these, there is an equivalence relation on a five element set with the number of ordered pairs in the relation given by the sum of the squares of the cell sizes. Thus the numbers that arise as the number of ordered pairs in some equivalence relation on A are 25, 17, 9, 11, 7, 5, and 13, and of these, 9 and 13 are listed.

The answer is C.

PART B

3 marks B1. Let G_1 and G_2 be groups, and let $G_1 \times G_2$ denote the direct product group. Prove that if G_1 and G_2 are abelian, then $G_1 \times G_2$ is abelian. For simplicity, use $*$ to denote the binary operation in each of G_1, G_2 and in $G_1 \times G_2$, so that for (a, b), $(c, d) \in G_1 \times G_2$, we have

$$(a, b) * (c, d) = (a * c, b * d).$$

(You do not need to prove that $G_1 \times G_2$ is a group.)

Solution: Let (a, b), $(c, d) \in G_1 \times G_2$. Then $(a, b) * (c, d) = (a * c, b * d)$. Since G_1 is abelian, we have $a * c = c * a$, and since G_2 is abelian, we have $b * d = d * b$. Thus $(a, b) * (c, d) = (a * c, b * d) = (c * a, d * b) = (c, d) * (a, b)$, and so $G_1 \times G_2$ is abelian.

4 marks B2. Let $\mathbb{R}$ denote the set of real numbers with the usual order relation. You may assume that the relation

$$P = \{\ ((x, y), (r, s)) \ \big|\ \text{either } x < r \text{ or else both } x = r \text{ and } y \le s\ \}$$

is a partial order relation on $\mathbb{R} \times \mathbb{R}$. Prove that P is a total order relation (that is, for each (x_1, y_1), $(x_2, y_2) \in \mathbb{R} \times \mathbb{R}$, $((x_1, y_1), (x_2, y_2)) \in P$ or $((x_2, y_2), (x_1, y_1)) \in P$).

Solution: Let (x_1, y_1), $(x_2, y_2) \in \mathbb{R} \times \mathbb{R}$. If $x_1 < x_2$, then $((x_1, y_1), (x_2, y_2)) \in P$, while if $x_2 < x_1$, then $((x_2, y_2), (x_1, y_1)) \in P$. Suppose that $x_1 = x_2$. Then either $y_1 \le y_2$, in which case $((x_1, y_1), (x_2, y_2)) \in P$, or else $y_2 \le y_1$, in which case $((x_2, y_2), (x_1, y_1)) \in P$.

4 marks B3. Prove that for all sets A and B, $A \cup (A \triangle B) = B \cup (A \triangle B)$.

Solution: $A \cup (A \triangle B) = A \cup ((A-B) \cup (A-B)) = (A \cup (A-B)) \cup (B-A) = A \cup (B-A) = A \cup (B-A) \cup (B \cap A) = A \cup B$. Similarly, $B \cup (A \triangle B) = B \cup ((A-B) \cup (A-B)) = (B \cup (B-A)) \cup (A-B) = B \cup (A-B) = B \cup (A-B) \cup (A \cap B) = B \cup A = A \cup B$. Thus $A \cup (A \triangle B) = A \cup B = A \cup B = B \cup (A \triangle B)$, as required.

Alternatively, we could show that $A \cup (A \triangle B) \subseteq B \cup (A \triangle B)$ and that $B \cup (A \triangle B) \subseteq A \cup (A \triangle B)$. Let $x \in A \cup (A \triangle B) = A \cup (A-B) \cup (B-A) = A \cup (B-A)$. If $x \in B$, then $x \in B \cup (A \triangle B)$. Suppose that $x \notin B$. Then since $x \in A$ or $x \in B-A$, we must have $x \in A$. But then $x \in A$ and $x \notin B$ means that $x \in A-B \subseteq A \triangle B$, so $x \in B \cup (A \triangle B)$ in this case as well. This completes the case-by-case analysis, and so we have proven that $x \in A \cup (A \triangle B)$ implies $x \in B \cup (A \triangle B)$, whence $A \cup (A \triangle B) \subseteq B \cup (A \triangle B)$. Similarly, let $x \in B \cup (A \triangle B) = B \cup (A-B) \cup (B-A) = B \cup (A-B)$. If $x \in A$, then $A \cup (A \triangle B)$. Suppose that $x \notin A$. Then since $x \in B$ or $x \in A-B$, we must have $x \in B$. But then we have $x \in B$ and $x \notin A$, so $x \in B-A \subseteq A \triangle B$, and again we have $x \in A \cup (A \triangle B)$. This proves that $x \in B \cup (A \triangle B)$ implies $x \in A \cup (A \triangle B)$, so $B \cup (A \triangle B) \subseteq A \cup (A \triangle B)$.

Since we have established that $A \cup (A \triangle B) \subseteq B \cup (A \triangle B)$ and $B \cup (A \triangle B) \subseteq A \cup (A \triangle B)$, we conclude that $A \cup (A \triangle B) = A \cup (A \triangle B)$.

4 marks B4. If G is a group of size 15 and H is a subgroup of G of size at least 6, determine the size of H. Provide reasons for your answer.

Solution: By Lagrange's theorem, the size of H is a divisor of the size of G. The positive divisors of 15 are $1, 3, 5, 15$. Since $|H| \geq 6$, the only possible value for $|H|$ is 15.

4 marks B5. Let G, H and K be groups, and let $f{:}\,G \to H$ and $g{:}\,H \to K$ be homomorphisms. Prove that $g \circ f$ is a homomorphism (for simplicity, use $*$ to denote the binary operation in each group).

Solution: Let $x, y \in G$. Then $g \circ f(x * y) = g(f(x * y)) = g(\, f(x) * f(y)\,) = g(f(x)) * g(f(y)) = g \circ f(x) * g \circ f(y)$. Thus $g \circ f$ is a homomorphism.

B6. Define a binary operation $*$ on $\mathbb{Q}$, the set of rational numbers, by the rule $a * b = 3ab$, where we are using the usual multiplication on $\mathbb{Q}$ to calculate $3ab$.

2 marks (a) Prove that $*$ is commutative.

Solution: Let $a, b \in \mathbb{Q}$. Then $a * b = 3ab = 3ba = b * a$, and so $*$ is commutative.

2 marks (b) Prove that $*$ is associative.

Solution: Let $a, b, c \in \mathbb{Q}$. Since $(a * b) * c = (3ab) * c = 3(3ab)c = 9abc$ and $a * (b * c) = a * (3bc) = 3a(3bc) = 9abc$, we have $(a * b) * c = a * (b * c)$ and so $*$ is associative.

3 marks (c) Prove that $*$ has an identity.

Solution: Suppose that $*$ has an identity element. Let e denote the identity of $*$, so that $a * e = e * a$ for each $a \in \mathbb{Q}$. Thus $3ae = a$ for every $a \in \mathbb{Q}$. In particular, this must hold for $a = 1$, so we have $3e = 1$ and thus $e = 1/3$. Since $(1/3) * a = (3)(1/3)a = a$ for every $a \in \mathbb{Q}$, and $*$ is commutative, so $a * (1/3) = (1/3) * a = a$ for every $a \in \mathbb{Q}$, it follows that $1/3$ is the identity for $*$.

4 marks B7. Let $a_0 = 2$, and for each integer $n \geq 1$, let $a_n = \sqrt{a_{n-1} + 1}$. Use mathematical induction to prove that $a_n \geq a_{n+1}$ for all integers $n \geq 0$.

Solution: Let $P(n){:}\,a_n \geq a_{n+1}$. We first show that $P(0){:}\,a_0 \geq a_1$ is true. We have $a_0 = 2 \geq \sqrt{3} = \sqrt{2+1} = \sqrt{a_0 + 1} = a_1$, and so $a_0 \geq a_1$. Thus $P(0)$ is true.

Now let $n \geq 0$ be any integer for which $P(n){:}\,a_n \geq a_{n+1}$ is true. We must show that $P(n+1){:}\,a_{n+1} \geq a_{n+2}$ is true. Now $a_{n+1} = \sqrt{a_n + 1}$ and $a_{n+2} = \sqrt{a_{n+1} + 1}$. Since $a_n \geq a_{n+1}$, we may add 1 to both sides to obtain $a_n + 1 \geq a_{n+1} + 1$. Since both $a_n + 1$ and $a_{n+1} + 1$ are positive, we may take square roots to obtain $a_{n+1} = \sqrt{a_n + 1} \geq \sqrt{a_{n+1} + 1} = a_{n+2}$, whence $P(n+1)$ is true. Thus for each integer $n \geq 0$, $P(n)$ implies $P(n+1)$ is true. Since $P(0)$ is true, and for each integer $n \geq 0$, $P(n)$ implies $P(n+1)$ is true, it follows from the principle of mathematical induction that $P(n){:}\,a_n \geq a_{n+1}$ is true for every integer $n \geq 0$.

2 marks B8. Let $A = \{\,1, 2, 3, 4\,\}$, let $B = \{\,a, b, c, d\,\}$ and let $f{:}\,A \to B$ be the function

$$f = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ a & b & c & c \end{smallmatrix}\right).$$

Write down the partition of A that is associated with the equivalence relation $\mathrm{Ker}(f)$. In other words, write down $A/\mathrm{Ker}(f)$.

Solution: The equivalence classes of $\mathrm{Ker}(f)$ are naturally in one-to-one correspondence with the elements of $\Im f$. More precisely, the equivalence classes of $\mathrm{Ker}(f)$ are the sets $f^{-1}(x)$ for $x \in \Im f$. Since $\Im f = \{\,a, b, c\,\}$, it follows that the partition of A if $\{\, f^{-1}(a), f^{-1}(b), f^{-1}(c)\,\} = \{\,\{\,1\,\}, \{\,2\,\}, \{\,3, 4\,\}\,\}$.

B9. Consider again the group $G = \{1, a, b, c, d, f, g, h\}$ with binary operation as given by the following table:

	1	a	b	c	d	f	g	h
1	1	a	b	c	d	f	g	h
a	a	g	h	b	f	1	c	d
b	b	h	1	f	g	c	d	a
c	c	b	f	d	a	g	h	1
d	d	f	g	a	b	h	1	c
f	f	1	c	g	h	d	a	b
g	g	c	d	h	1	a	b	f
h	h	d	a	1	c	b	f	g

3 marks (a) Establish whether or not the subset $H = \{1, a, g, b\}$ is a subgroup of G. Explain your answer.

Solution: Since $ag = c \notin H$, H is not closed under the binary operation and so H is not a subgroup of G.

3 marks (b) Compute $\langle g \rangle$, the subgroup generated by g. Show your work.

Solution: Since $g^2 = b$, $g^3 = gg^2 = gb = d$, and $g^4 = (g^2)^2 = b^2 = 1$, we see that $\langle g \rangle = \{1, g, b, d\}$.

2 marks (c) You may assume that $H = \{1, b\}$ is a subgroup of G. Find the left coset cH.

Solution: By definition, $cH = \{cx \mid x \in H\} = \{c, cb\} = \{c, f\}$.

S. Rankin MW 2-4

D. Christensen MW 4-6 or MTWTh 12-1

Circle your instructor's name ____________________ Student's Name (**Print**) ____________________

Student's Signature ____________________ Student Number ____________________

THE UNIVERSITY OF WESTERN ONTARIO
LONDON CANADA
DEPARTMENT OF MATHEMATICS

Mathematics 222a Final Exam

December 16, 2001 2:00–5:00 p.m.

INSTRUCTIONS

1. BE SURE TO FILL IN THE TOP OF THIS PAGE CORRECTLY.

2. THE EXAM CONSISTS OF TWO PARTS (A AND B).

3. The first part of the exam (Part A: Questions A1–A30) is MULTIPLE CHOICE. This part is to be answered on the SCANTRON answer sheet, and at the same time, circle your selected answer on the question sheet.

   The second part (Part B) has questions to be answered in the space provided. Be sure to SHOW ALL OF YOUR WORK, and try to place your solution in the space provided for that purpose.

4. Print your name and your instructor's name on the SCANTRON answer sheet. Sign the answer sheet, and mark your student number and section on the answer sheet. USE A PENCIL to mark your answers to Questions A1–A30 in the left column of the SCANTRON answer sheet.

5. There are two blank pages at the back of this booklet. These may be torn off and used for scrap paper as the need arises. We suggest that you keep one blank page attached in case you need extra space to answer a question. Do not tear any other pages from the booklet.

6. Questions start on Page 1 and continue to Page 9. BE SURE YOUR BOOKLET IS COMPLETE.

7. CALCULATORS ARE NOT PERMITTED.

8. TOTAL MARKS = 100.

FOR GRADING ONLY

PAGE	MARK
1	
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