Student's Name [print] Student Number

Mathematics 222a First Midterm Exam, Exam Code 111 **October 18, 2002**

7:00–9:30 p.m.

Instructions: Print your name and your instructor's name on the SCANTRON answer sheet. Sign the SCANTRON answer sheet, and mark your student number and section on the SCANTRON answer sheet. Use a PENCIL to mark your answers to questions 1–22 on the SCANTRON answer sheet, and as well, circle your answers on the question sheet.

2 marks A1. Let $P(x)$ and $Q(x)$ be predicates with domain S. The **negation** of the statement "For all $x \in S$ ($P(x)$ and $\neg Q(x)$)" is logically equivalent to:

(i) There exists $x \in S$ (($\neg P(x)$) or $Q(x)$).
(ii) For all $x \in S$ ($\neg$($P(x)$ and $Q(x)$)).
(iii) There exists $x \in S$ (($\neg P(x)$) and ($\neg Q(x)$)).
(iv) For all $x \in S$ (($\neg P(x)$) or ($\neg Q(x)$)).

A: (i)	B: (ii)	C: (iii)	D: (iv)	E: None of A, B, C, D

Solution: The negation of the statement "For all $x \in S$ ($P(x)$ and $\neg Q(x)$)" is logically equivalent to "There exists $x \in S$ ($\neg$($P(x)$ and ($\neg Q(x)$)))", which in turn is logically equivalent to "There exists $x \in S$ (($\neg P(x)$) or $Q(x)$).

Thus the answer is A.

2 marks A2. In how many ways can 7 married couples be seated in a circle if each couple must sit side by side?

A: 7^2	B: $6!2^7$	C: $6!$	D: $6!7^2$	E: $7!2^7$

Solution: Think of each couple as a "person", and place these 7 "people" around the table. This can be done in 6! ways. For each such assignment, each "person" can be replaced by seating the two members of the couple, in either of 2 ways. Thus there are 2^7 ways to seat the 7 couples for each allocation of 7 pairs of chairs, and there are 6! ways to allocate the 7 pairs, for a total of $2^7 6!$ ways to seat the 7 married couples.

The answer is B.

2 marks A3. Let $A = \{ 1, \{ 1 \}, \{ 2 \}, \{ 1, 2 \} \}$. Exactly which of the following statements are true?

(i) $1 \in A$ (ii) $2 \in A$. (iii) $\{1\} \in A$ (iv) $\{1\} \subseteq A$. (v) $\{1, 2\} \subseteq A$.

A: (i)–(v)	B: All but (ii)	C: All but (v)	D: (i), (iii), (iv)	E: (i), (iii), (v)

Solution:
(i) is true, since 1 is an element of A.
(ii) is false, since 2 is not an element of A.
(iii) is true, since $\{ 1\{$ is an element of A.

(iv) is true, since 1 is an element of A, and so $\{1\} \subseteq A$.
(v) is false, since 2 is not an element of A.
The (i), (iii) and (iv) are true, while (ii) and (v) are false. The answer is D.

2 marks A4. Let $A = \{1, 2, 4, 5\}$, $B = \{1, 3, 4, 8\}$ and $C = \{1, 2, 3, 5, 7\}$. Exactly which of the following statements are true?

(i) $A - (B \cap C) = \{2, 4, 5\}$.
(ii) $B \cap (A - C) = \{4\}$.
(iii) $(A \cup B) - (A \cup C) = \{4, 8\}$.
(iv) $(A - B) - C = \varnothing$.

A: (i), (iii)	B: (i), (ii), (iv)	C: (i), (ii), (iii)	D: (i), (iii), (iv)	E: (iii), (iv)

Solution: (i) Since $B \cap C = \{1, 3\}$, we have $A - (B \cap C) = \{1, 2, 4, 5\} - \{1, 3\} = \{2, 4, 5\}$. Thus (i) is true.
(ii) Since $A - C = \{4\}$, we have $B \cap (A - C) = \{1, 3, 4, 8\} \cap \{4\} = \{4\}$. Thus (ii) is true.
(iii) We have $(A \cup B) - (A \cup C) = B - (A \cup C)$, and $A \cup C = \{1, 2, 3, 4, 5, 7\}$, so $(A \cup B) - (A \cup C) = \{1, 3, 4, 8\} - \{1, 2, 3, 4, 5, 7\} = \{8\}$. Thus (iii) is false.
(iv) $A - B = \{2, 5\}$, so $(A - B) - C = \{2, 5\} - \{1, 2, 3, 5, 7\} = \varnothing$. Thus (iv) is true.
Since (i), (ii), and (iv) are true, while (iii) is false, the answer is B.

2 marks A5. Given a set S, exactly which of the following are true for all subsets A, B and C of S?

(i) $A \subseteq B$ if and only if $A \cap B = A$.
(ii) $A \subseteq B$ if and only if $B - A = \varnothing$.
(iii) $A \triangle B = A \triangle C$ implies $B = C$.
(iv) $A - B^c = A \cup B$.

A: All of them	B: (i), (ii), (iii)	C: (i), (ii), (iv)
D: (ii), (iv)	E: (i), (iii)	

Solution:
(i) is true by a result in the text.
(ii) false, since by a result in the text, $A \subseteq B$ if and only if $A - B = \varnothing$, and it is certainly not true that $A - B = \varnothing$ if and only if $B - A = \varnothing$. Consider for example $A = \varnothing$ and $B = \{1\}$.
(iii) is true by a result in the text.
(iv) is false. We have $A - B^c = A \cap (B^c)^c = A \cap B$. Since in general, $A \cap B \neq A \cup B$ (for example, $A = \{1\}$ and $B = \{2\}$ have $A \cap B = \varnothing$ and $A \cup B = \{1, 2\}$), (iv) is false.
Thus (i) and (iii) are true, while (ii) and (iv) are false. The answer is E.

2 marks A6. Exactly which of the following are true for all subsets A, B and C of a given set S?

(i) $(A - B) \cup C = (C - B) \cup A$.
(ii) $(A - B) \cap C = (C - B) \cap A$.
(iii) $(A^c \cap B^c)^c \cap C^c = (A \cup B) - C$.
(iv) $A \subseteq B$ if and only if $A^c \cup B = S$.

A: (ii), (iii)	B: (ii), (iv)	C: (ii), (iii), (iv)	D: (i), (iii), (iv)	E: (iii), (iv)

Solution:

i) False. $(A-B)\cup C=(A\cap B^c)\cup C=(A\cup C)\cap(B^c\cup C)$, $(C-B)\cup A=(C\cap B^c)\cup A=(A\cup C)\cap(B^c\cup A)$. For a counterexample, we must have $B^c\cup C\neq B^c\cup A$. Try extremes, say $C=\varnothing$. In this case, the equation reduces to $A-B=A$, so take $B=A\neq\varnothing$, say $A=\{1\}=B$. Then $(A-B)\cup C=\varnothing$, while $(C-B)\cup A=\{1\}$.

ii) True, since $(A-B)\cap C=A\cap B^c\cap C=A\cap(C\cap B^c)=A\cap(C-B)$.

iii) True, since $(A\cup B)\cap(A^c\cup C^c)\cap(B^c\cup C^c)=(A\cup B)\cap[(A^c\cap B^c)\cup C^c]=(A\cup B)\cap C^c=(A\cup B)-C$.

iv) True, since $A\subseteq B\iff A\cap B=A=A\cap S=A\cap(B\cup B^c)=(A\cap B)\cup(A\cap B^c)\iff A\cap B^c\subseteq A\cap B\iff A\cap B^c=(A\cap B)\cap(A\cap B^c)=A\cap B\cap B^c=\varnothing\iff A^c\cup B=S$.

Thus (ii), (iii) and (iv) are true while (i) is false, and so the answer is C.

2 marks A7. Which of the following are true for all sets A, B and C?

(i) $A\times(B\cup C)=(A\times B)\cup(A\times C)$; (iii) $A\times\varnothing=A$;

(ii) $A\cup(B\times C)=(A\cup B)\times(A\cup C)$; (iv) $A\times B=B\times A$

A: (ii), (iv)	B: (i), (ii)	C: (ii), (iii), (iv)	D: (i) only	E: (i), (iv)

Solution: (i) is true. For if $(a,r)\in A\times(B\cup C)$, then $a\in A$ and $r\in B\cup C$, so either $r\in B$ in which case $(a,r)\in A\times B$ and thus $(a,r)\in(A\times B)\cup(A\times C)$, or else $r\in C$, in which case $(a,r)\in A\times C$ and thus $(a,r)\in(A\times B)\cup(A\times C)$. Thus $(a,r)\in A\times(B\cup C)$ implies that $(a,r)\in(A\times B)\cup(A\times C)$, so $A\times(B\cup C)\subseteq(A\times B)\cup(A\times C)$. Conversely, if $(a,r)\in(A\times B)\cup(A\times C)$, then either $(a,r)\in A\times B$, in which case $(a,r)\in A\times(B\cup C)$, or else $(a,r)\in A\times C$, in which case we also have $(a,r)\in A\times(B\cup C)$. Thus $(a,r)\in(A\times B)\cup(A\times C)$ implies that $(a,r)\in A\times(B\cup C)$, so $(A\times B)\cup(A\times C)\subseteq A\times(B\cup C)$.

Thus we have $A\times(B\cup C)\subseteq(A\times B)\cup(A\times C)\subseteq A\times(B\cup C)$, so $A\times(B\cup C=(A\times B)\cup(A\times C)$.

(ii) is false, since if $A=\{1\}$, $B=\varnothing$, and $C=\{2\}$, then $B\times C=\varnothing$ and so $A\cup(B\times C)=A\cup\varnothing=A$, while $(A\cup B)\times(A\cup C)=\{1\}\times\{1,2\}\neq A$.

(iii) is false, since $A\times\varnothing=\varnothing$, so for $A\neq\varnothing$, say $A=\{1\}$, we have $A\times\varnothing\neq A$.

(iv) is false, since for $A=\{1\}$ and $B=\{2\}$, we have $A\times B=\{(1,2)\}$ and $B\times A=\{(2,1)\}$. Since $(1,2)\neq(2,1)$, it follows that $A\times B\neq B\times A$ in this case.

Thus only (i) is true, so the answer is D.

2 marks A8. For exactly which of the predicates $P(n)$ shown below is the statement "For each $n\geq 0$, $P(n)$ implies $P(n+1)$" true?

(i) $n<n+1$.

(ii) $n+1<n$.

(iii) $n!=n^2+1$.

A: (i), (ii)	B: (i), (iii)	C: (i)	D: (iii)	E: None of A, B, C, D

Solution:

(i) $P(n):\ n<n+1$ is true for every integer $n\geq 0$, so $P(n)$ implies $P(n+1)$ is true for every integer $n\geq 0$.

(ii) $P(n):\ n+1<n$ is false for every integer $n\geq 0$, so $P(n)$ implies $P(n+1)$ is true for every integer $n\geq 0$.

(iii) $P(0)$ is the assertion that $0!=1$, which is true, while $P(1)$ is the assertion that $1=2$, which is false. It follows that $P(0)$ does not imply $P(1)$. (We remark that $P(2)$ is the assertion that $2=6$, which is false, and $P(3)$ is the assertion that $6=10$, which is false. Finally, since for $n\geq 4$, $n!\geq(n)(n-1)(n-2)=(n^2-n)(n-2)\geq(n^2-n)(2)=n^2+(n^2-2n)=n^2+(n^2-2n+1)-1=n^2+(n-1)^2-1\geq n^2+(3)^2-1=n^2+8>n^2+1$, we see that $P(n)$ is false for every $n\geq 4$. Thus $P(0)$ is true, while $P(n)$ is false for $n\geq 1$.)

Thus (i) and (ii) lead to true statements, while (iii) does not. The answer is A.

2 *marks* A9. Exactly which of the following statements are true for all sets A, B, and C?

(i) $(A \cup B) - (B \cup C) = A - C$.
(ii) If $A \subseteq C$, then $(A \cap B) \cup C = C$.
(iii) $A \cup (B \cap C) = (A \cup B) \cap C$.

A: (i), (iii)	B: (i), (ii)	C: (i), (ii), (iii)	D: (i)	E: (ii)

Solution:
(i) False. In fact, we have $(A \cup B) - (B \cup C) = (A - (B \cup C)) \cup (B - (B \cup C)) = (A - (B \cup C)) \cup \varnothing = A - (B \cup C)$. If we take $A = B = \{1\}$ and $C = \{2\}$, then $A - C = A$, while $A - (B \cup C) = A - (A \cup C) = \varnothing$. Thus in this example, $A - C$ is not empty, while $(A \cup B) - (B \cup C)$ is empty, so $(A \cup B) - (B \cup C) \neq A - C$ in this case.
(ii) True. Let A, B and C be sets with $B \subseteq C$, Then $A \cap B \subseteq A \subseteq C$, and so $C = C \cup (A \cap B)$.
(iii) False. For if it were true, then for $B = \varnothing$, we would have $A = A \cap C$ for any sets A, C. Take A non-empty and $C = \varnothing$, say $A = \{1\}$, to find that $A \cup (B \cap C) = \{1\}$, while $(A \cup B) \cap C = \varnothing$.
Since only (ii) is true, the answer is E.

2 *marks* A10. For each $n \in \mathbb{N}$, let $A_n = [-n-1, n+1] = \{ x \in \mathbb{R} \mid -n-1 \le x \le n+1 \}$. Then which of the following is true? (Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$.)

A: $\bigcap_{n\in\mathbb{N}} A_n = (-1, 1)$ and $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R} - \mathbb{Z}$;
B: $\bigcap_{n\in\mathbb{N}} A_n = \{0\}$ and $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}$;
C: $\bigcap_{n\in\mathbb{N}} A_n = \varnothing$ and $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}$;
D: $\bigcap_{n\in\mathbb{N}} A_n = [-1, 1]$ and $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R} - \mathbb{Z}$.
E: None of A, B, C, D

Solution: Note that $A_0 = [-1, 1] \subseteq A_1 = [-2, 2] \subseteq \cdots$. Thus $\bigcap_{n\in\mathbb{N}} A_n = (-1, 1) = A_0 = [-1, 1]$, while $\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}$. Thus the correct answer is E.

2 *marks* A11. If $A = \{1, 2, 3, 4, 5, 6, 7\}$, then the number of subsets of A which contain at least one odd integer is:

A: 2^4	B: 99	C: 120	D: 180	E: 240

Solution: We choose to count the total number of subsets of A, and subtract the number of those subsets which do not contain any odd integers. Since to form a subset of A which does not contain an odd integer is to choose a subset of $\{2, 4, 6\}$, the number of subsets of A which do not contain any odd integers is 2^3. Thus the number of subsets of A which contain at least one odd integer is$2^7 - 2^3 = 2^3(2^4 - 1) = (8)(15) = 120$. The answer is C.

2 *marks* A12. The coefficient of x^6y^9 in the binomial expansion of $(3x^2 - 2y^3)^5$ is:

A: $\binom{5}{3}3^2(-2)^3$	B: $\binom{5}{3}$	C: -720	D: -120	E: None of A, B, C, D

Solution: Since $(x^2)^i(y^3)^{5-i}$ is not equal to x^6y^9 for any value of i, no term in the binomial expansion of $(3x^2 - 2y^3)^5$ is of the form a constant times x^6y^9. Thus the coefficient of x^6y^9 in the binomial expansion of $(3x^2 - 2y^3)^5$ is zero. The answer is E.

2 marks A13. The sum of the coefficients of the multinomial expansion of $(2x - 3y + z - w)^{15}$ is:

A: $15!/5!3!4!4!$	B: 0	C: 1
D: -1	E: None of A, B, C, D	

Solution: Put $x = y = z = w = 1$ in $(2x - 3y + z - w)^{15}$. The result is -1. The answer is D.

2 marks A14. Let $A = \{1, 2, 3\}$ and $B = \{10, 11\}$. How many relations from A to B contain at most one of the pairs $(1, 10)$ and $(1, 11)$?

A: $3(2^4)$	B: $2^3 - 2$	C: 2^6	D: 2^4	E: $2^6 - 2$

Solution: We count those that contain exactly one of the pairs, and those that contain neither of the pairs and sum the results. First, let us determine the number of subsets of $A \times B$ that contain $(1, 10)$ and not $(1, 11)$. There are $(3)(2) = 6$ elements in $A \times B$. We must take $(1, 10)$ and not $(1, 11)$, so remove $(1, 11)$ and take $(1, 10)$. Now we may take any subset of the remaining 4 ordered pairs, so there are 2^4 such relations. Similarly, there will be 2^4 relations that contain $(1, 11)$ and not $(1, 10)$. Finally, to determine the number of relations that contain neither, we observe that this just means that our relation is a subset of the four pairs different from $(1, 10)$ and $(1, 11)$, so there are 2^4 of these relations. Thus altogether there are $2^4 + 2^4 + 2^4 = 3(2^4)$ such relations.

The answer is A.

2 marks A15. Exactly which of the following relations on $\mathbb{N}$ are reflexive?

(i) $R = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a = b + 1\}$.

(ii) $S = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a \le 2b\}$.

(iii) $T = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a^2 + b^2 > 0\}$.

A: R	B: S	C: T	D: $S,\ T$	E: $R,\ S,\ T$

Solution:
(i) R is not reflexive, for if $a \in \mathbb{N}$ satisfies $a = a + 1$, then $0 = 1$, which is not true. Thus for every $a \in \mathbb{N}$, $(a, a) \notin R$.
(ii) S is reflexive, since for any $a \in \mathbb{N}$, $a \ge 0$, and so multiplying the inequality $1 \le 2$ by a yields $a \le 2a$, whence $(a, a) \in S$ for every $a \in \mathbb{N}$.
(iii) T is not reflexive, for $0^2 + 0^2 = 0$, so $(0, 0) \notin T$.

Thus neither R nor T are reflexive, while S is reflexive.

The answer is B.

2 marks A16. Exactly which of the following relations on $\mathbb{N}$ are transitive?

(i) $R = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a = b + 1\}$.

(ii) $S = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a \le 2b\}$.

(iii) $T = \{(a, b) \in \mathbb{N} \times \mathbb{N} \mid a^2 + b^2 > 0\}$.

A: R	B: S	C: T	D: All of them	E: None of A, B, C, D

Solution:
(i) R is not transitive, for $(2, 1) \in R$ and $(1, 0) \in R$, but $(2, 0) \notin R$.
(ii) S is not transitive, since $(4, 2) \in S$ and $(2, 1) \in S$, but $(4, 1) \notin S$.
(iii) T is not transitive. For $(0, 1), (1, 0) \in T$, but $(0, 0) \notin T$.

Thus not a one of the three relations is transitive.

The answer is E.

2 marks A17. In how many ways can the letters in the word RATIONALIZATION be arranged if there must be at least four other letters between the two N's?

A: $\frac{1}{4!}\frac{15!}{3!3!2!2!2!}$	B: $\frac{13!}{2!2!3!3!}$	C: $\binom{11}{9}\frac{13!}{3!3!2!2!}$	D: 13!/4	E: None of A, B, C, D

Solution: There are 15 letters in all. Think of the pattern _ _.... N _ _ _ _ _ ... _ N _ ... _ and let x_1 denote the number of blanks to place before the first N, let x_2 denote the number of blanks in excess of 4 to place between the two N's, and let x_3 denote the number of blanks to place to the right of the second N. Then $x_1 + x_2 + x_3 = 15 - 2 - 4 = 9$, and $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$. The number ways of placing the 13 blanks is thus the number of solutions to this constrained equation, which we know to be $\binom{9+2}{9}$. For each such arrangement of blanks, there are 13!/(3!2!3!2!) ways to arrange the 13 non-N letters. Thus there are $\binom{11}{9}\frac{13!}{3!3!2!2!}$ such arrangements. The answer is C.

2 marks A18. In how many ways can the letters in the word RATIONALIZATION be arranged if all of the I's must be grouped together?

A: 13!	B: 13!/(2!2!2!3!)	C: $2^{13}/3!$	D: 2^{11}	E: 13!/3!

Solution: 13!/(2!2!2!3!), because we may clump the 3 I's together to form one new token, for a total of 13, III-1, R-1, A-3, T-2, O-2, N-2, L-1, and Z-1.

2 marks A19. Which of the following expressions have the same value as $\binom{m+1}{n}$ for all positive integers m and n with $m \geq n$?

(i) $\binom{m+1}{m-n}$

(ii) $\binom{m}{n} + \binom{m}{n-1}$

(iii) $\frac{m+1}{n}\binom{m}{n-1}$

A: (i) only
B: (i) and (ii) only
C: (ii) and (iii) only
D: All of (i), (ii), (iii)
E: (i) and (iii) only

Solution:
(i) $\binom{m+1}{m-n} = \frac{(m+1)!}{(m-n)!(m+1-m+n)!} = \frac{(m+1)!}{(m+1-(n+1))!(n+1)!} = \binom{m+1}{n+1} \neq \binom{m+1}{n}$ (unless $m = 2n+1$). Thus in general, (i) does not have the same value as $\binom{m+1}{n}$.
(ii) $\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n}$ by Pascal's identity.
(iii) $\frac{m+1}{n}\binom{m}{n-1} = \frac{m+1}{n}\frac{m!}{(n-1)!(m-n+1)!} = \frac{(m+1)!}{n!(m+1-n)!} = \binom{m+1}{n}$, so (iii) has the same value as $\binom{m+1}{n}$.

Thus only (ii) and (iii) have the same value as $\binom{m+1}{n}$, so the answer is C.

2 marks A20. In how many ways can one distribute 6 identical balls into 4 distinct boxes labelled A, B, C, and D, if boxes A and B are not to be left empty?

A: 4^6	B: 2^6	C: $\binom{10}{3}$	D: 48	E: 35

Solution: Put one ball into box A and one ball into box B. We are left with the problem of determining the number of ways of putting 4 identical balls into 4 distinct labelled boxes, which is the number of solutions in integers for the equation $x_A + x_B + x_C + x_D = 4$ constrained by the requirements that $x_A, x_B, x_C, x_D \geq 0$. This is $\dfrac{(4+4-1)!}{4!(4-1)!} = \dbinom{7}{4} = \dbinom{7}{3} = (7)(6)(5)/(6) = 35$. The answer is E.

2 marks A21. While on holiday I buy postcards to send home. If I buy 3 copies of each of 4 different postcards, how many ways can I distribute these cards among 6 friends? There are no other restrictions — any friend may get any number of postcards, including multiple copies of the same card.

A: $\binom{6}{3}^4$	B: $\binom{4}{3}^6$	C: $\binom{9}{4}^3$	D: $\binom{8}{3}^4$	E: $\binom{6}{3}^6$

Solution: For any one of the 4 different types of postcard, we distribute the three copies of the card to the 6 friends. The number of ways to do this is equal to the number of solutions in integers to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 3$ subject to $x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$. This is $\frac{(3+(6-1))!}{3!(6-1)!} = \binom{8}{3} = \binom{8}{5}$. When this is done for each card, we have $\binom{8}{3}^4 = \binom{8}{5}^4$ possible outcomes.

The answer is D.

2 marks A22. While on holiday I buy postcards to send home. If I buy 3 copies of each of 4 different postcards, how many ways can I distribute these cards among 6 friends if no friend may get more than one copy of any one card?

A: $\binom{6}{3}^4$	B: $\binom{4}{3}^6$	C: $\binom{9}{4}^3$	D: $\binom{8}{3}^4$	E: $\binom{6}{3}^6$

Solution: For any one of the 4 different types of postcard, we distribute the three copies of the card to the 6 friends by choosing a subset of size 3 from the 6. The number of ways to do this is $\binom{6}{3}$. When this is done for each type of card, we have $\binom{6}{3}^4$ possible outcomes. The answer is A.

4 marks B1. Prove using induction that $\displaystyle\sum_{i=1}^{n} \frac{1}{(4i-3)(4i+1)} = \frac{n}{4n+1}$ for all integers $n \geq 1$.

Solution: Let $P(n)$ denote the predicate "$\displaystyle\sum_{i=1}^{n} \frac{1}{(4i-3)(4i+1)} = \frac{n}{4n+1}$". Consider $P(1)$, which is the assertion that $\displaystyle\sum_{i=1}^{1} \frac{1}{(4i-3)(4i+1)} = \frac{1}{4+1}$. Since $\displaystyle\sum_{i=1}^{1} \frac{1}{(4i-3)(4i+1)} = \frac{1}{(1)(5)} = \frac{1}{4+1}$, $P(1)$ is true.

Suppose now that $n \geq 1$ is an integer for which $P(n)$ is true, so we know that $\displaystyle\sum_{i=1}^{n} \frac{1}{(4i-3)(4i+1)} =$

$\frac{n}{4n+1}$. Consider $P(n+1)$, which is the assertion that $\displaystyle\sum_{i=1}^{n+1} \frac{1}{(4i-3)(4i+1)} = \frac{n+1}{4(n+1)+1}$. We have

$$\begin{aligned}
\sum_{i=1}^{n+1} \frac{1}{(4i-3)(4i+1)} &= \left(\sum_{i=1}^{n} \frac{1}{(4i-3)(4i+1)} \right) + \frac{1}{(4(n+1)-3)(4(n+1)+1)} \\
&= \frac{n}{4n+1} + \frac{1}{(4(n+1)-3)(4(n+1)+1)} \\
&= \frac{n}{4n+1} + \frac{1}{(4n+1)(4n+5)} \\
&= \frac{n(4n+5)+1}{(4n+1)(4n+5)} \\
&= \frac{(4n^2+5n+1)}{(4n+1)(4n+5)} \\
&= \frac{(4n+1)(n+1)}{(4n+1)(4n+5)} \\
&= \frac{n+1}{4n+5}.
\end{aligned}$$

Thus $P(n+1)$ is true. This proves that for all integers $n \geq 1$, $P(n)$ implies $P(n+1)$. It follows now by the first principle of mathematical induction that $P(n)$ is true for all integers $n \geq 1$.

4 marks B2. Prove using induction that $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$ for every integer $n \geq 1$.

Solution: Let $P(n)$ denote the predicate "$\sum_{i=1}^{n} i(i!) = (n+1)! - 1$". Consider $P(1)$, which is the assertion that $\sum_{i=1}^{1} i(i!) = (1+1)! - 1$. Since $\sum_{i=1}^{1} i(i!) = 1(1)! = 1$ and $(1+1)! - 1 = 2 - 1 = 1$, $P(1)$ is true.

Suppose now that $n \geq 1$ is an integer for which $P(n)$ is true, so we know that $\sum_{i=1}^{n} i(i!) = (n+1)! - 1$. Consider $P(n+1)$, which is the assertion that $\sum_{i=1}^{n+1} i(i!) = (n+2)! - 1$. We have $\sum_{i=1}^{n+1} i(i!) = \left(\displaystyle\sum_{i=1}^{n} i(i!) \right) + (n+1)(n+1)! = (n+1)! - 1 + (n+1)(n+1)! = (n+1)!(1+n+1) - 1 = (n+2)! - 1$, so $P(n+1)$ is true. This proves that for all integers $n \geq 1$, $P(n)$ implies $P(n+1)$. It follows now by the first principle of mathematical induction that $P(n)$ is true for all integers $n \geq 1$.

B3. Let $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7\}$, $C = \{5,6,7,8,9\}$ and $D = \{1,3,5,7,9\}$. Find:

2 marks (a) $A \triangle D$;

Solution: $A \triangle D = (A - D) \cup (D - A) = \{2,4\} \cup \{7,9\} = \{2,4,7,9\}$.

2 marks (b) $(A - D) - B$;

Solution: $(A - D) - B = \{2,4\} - \{4,5,6,7\} = \{2\}$.

4 marks B4. Prove that if A, B and C are sets with $A \subseteq C$, then $A \cup (B \cap C) = (A \cup B) \cap C$. (Explain fully; a Venn diagram does not constitute a proof.)

Solution: Let A, B and C be sets with $A \subseteq C$. By the distributive law, we obtain $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. But since $A \subseteq C$, we have $A \cup C = C$, whence $A \cup (B \cap C) = (A \cup B) \cap C$.

4 marks B5. Find the number of integers n with $1000 \leq n \leq 9999$ and which contain the digit 3 exactly once. Establish your answer.

Solution: The number of 4-digit integers which contain 3 exactly once, and for which 3 occurs as the first digit is 9^3, while the number of 4-digit integers which contain 3 exactly once, and 3 is not the first digit is $\binom{3}{1}(8)(9^2)$. The total number of 4-digit integers in which 3 appears exactly once is therefore $9^3 + (3)(8)9^2 = 81(9+24) = (81)(33) = 243 + 2430 = 2673$.

B6. How many solutions are there for the equation $w + x + y + z = 12$ if:

2 marks (a) w, x, y, and z are nonnegative integers?

Solution: $\binom{12+3}{12} = (5)(7)(13) = 455$.

2 marks (b) $w \geq -2$, $x \geq -1$, $y \geq 0$, and $z \geq 1$?

Solution: $w+x+y+z = 12$ with $w \geq -2$, $x \geq -1$, $y \geq 0$, and $z \geq 1$ if and only if $(w+2)+(x+1)+(y)+(z-1) = 12+2+1-1 = 14$, where $w+2 \geq 0$, $x+1 \geq 0$, $y \geq 0$ and $z-1 \geq 0$. Thus the number of solutions is $\binom{14+3}{14} = \binom{17}{3} = (17)(8)(5) = 680$.

B7. In each case, either give an example of a relation R on the set $A = \{1, 2, 3\}$ with the indicated properties or explain why no such relation exists.

2 marks (a) reflexive, antisymmetric, not symmetric and not transitive.

Solution: Consider $R = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$. Since $1_A \subseteq R$, R is reflexive. Since $(1,2) \in R$ but $(2,1) \notin R$, R is not symmetric. Since $R \cap R^{-1} = \{(1,1), (2,2), (3,3)\} \subseteq 1_A$, R is antisymmetric. Finally, $(1,2) \in R$ and $(2,3) \in R$, but $(1,3) \notin R$, so R is not transitive.

2 marks (b) symmetric, not antisymmetric, transitive, and not reflexive.

Solution: Since a relation R is symmetric and antisymmetric if and only if R is a subset of 1_A, and we require R to be symmetric, we must have at least one non-diagonal pair in R. Since R is not to be reflexive, we must not have all diagonal pairs in R. Let us try to keep R as small as possible. Put $(1,2)$ in R. Since R must be symmetric, we must then also put $(2,1)$ in R. Then transitivity would require that we put $(1,1)$ and $(2,2)$ in R. Let us try $R = \{(1,1), (1,2), (2,1), (2,2)\}$. We have $R = R^{-1}$, so R is symmetric. As well, since $R \cap R^{-1} = R \not\subseteq 1_A$, R is not antisymmetric. Next, since $R \circ R = \{(1,1), (1,2), (2,1), (2,2)\} = R$, R is transitive. Finally, since $(3,3) \notin R$, R is not reflexive.

4 marks B8. The number 21^{94} has 121 digits and is approximately 1.943×10^{124}. The last digit is 1. What is the second-last digit? Establish your answer. (Suggestion: use the binomial theorem.)

Solution:

$$\begin{aligned}
21^{94} = (20+1)^{94} &= \sum_{i=0}^{94} \binom{94}{i} 20^i 1^{94-i} \\
&= \sum_{i=0}^{94} \binom{94}{i} 20^i = \binom{94}{0} 20^0 + \binom{94}{1} 20^1 + 20^2 \left(\sum_{i=2}^{94} \binom{94}{i} 20^{i-2} \right) \\
&= 1 + (94)(20) + 100(4) \left(\sum_{i=2}^{94} \binom{94}{i} 20^{i-2} \right),
\end{aligned}$$

so the last two digits of 21^{94} are the last two digits of $1 + 1880 = 1881$. The second last digit is therefore 8. In fact, Maple tells us that

$$\begin{aligned}
21^{94} = 19{,}436, 304, 976, 371, 000, 376, 663, 555, 858, 504, \\
624, 794, 297, 280, 482, 452, 401, 156, 528, 525, \\
632, 657, 830, 346, 207, 774, 356, 881, 933, 189, \\
358, 382, 039, 697, 247, 332, 021, 652, 379, 862, 281.
\end{aligned}$$

4 marks B9. In how many ways can the letters in the word INTERSESSION be arranged if exactly one of the two E's is to be situated between two S's? (There may be other letters in addition to the E between two S's, so for example, INTERSEISSON is such an arrangement, but INTRSESEISON and INTRSEESISON are not.) Establish your answer.

Solution: –S–S–S– Choose one of the two internal stretches in which to place an E, and choose one of the two external stretches in which to place an E. No matter which choice is made, the result is to have 6 positions where blanks may be inserted, upon which we will arrange the 7 remaining letters, which can be done in $7!/(2!2!)$ ways. The number of ways to position the blanks is $(7+5)!/7!5! = \binom{12}{5}$, so the total number of arrangments is $4\binom{12}{5}7!/(2!2!)$.

Alternatively, choose 5 blanks from a row of 12. Place the 3 S's and 2 E's on the selected 5 blanks. There are 4 ways to do this: ESSES, SESSE, ESESS, SSESE. For each such placement, we arrange the remaining 7 letters, which can be done in $\frac{7!}{(2!2!)}$ ways. Altogether, there are $4\binom{12}{5}\frac{7!}{(2!2!)}$ such arrangements.