Student's Name [print] ______________________ Student Number ______________

Mathematics 222a Quiz 1 **CODE 222** **October 8, 2002**

Instructions: Print your name and student number at the top of this question sheet. Print your name and your instructor's name on the answer sheet. Sign the answer sheet, and mark your student number and section on the answer sheet. The **code** for this question sheet is shown above. Mark this number in the "code" box on the answer sheet. *Both sheets must be handed in at the end of the quiz!* Mark your answers to all questions (1–15) in the left column of the answer sheet.

Calculators are not permitted.

1. [1 mark] How many subsets of $\{1,2,3,4,5\}$ contain 1 or 3 or both?

A: 24	B: 26	C: 28	D: 30	E: 32

Solution: Count the number of subsets of $\{1,2,3,4,5\}$ that contain 1 but not 3, then count the number of subsets of $\{1,2,3,4,5\}$ that contain 3 but not 1, then count the number of subsets of $\{1,2,3,4,5\}$ that contain both 1 and 3. These are, respectively, the number of subsets of $\{2,4,5\}$ (each to have 1 put in), the number of subsets of the same set $\{2,4,5\}$, this time each to have 3 put in, and the number of subsets of the same set $\{2,4,5\}$, this time each is to have 1 and 3 put in. Thus the answer is $3(2^3) = 24$.

Alternatively, we could let A denote the set of subsets of $\{1,2,3,4,5\}$ which contain 1 and let B denote the set of those which contain 3. Then $A \cup B$ is the set of all subsets of $\{1,2,3,4,5\}$ which contain 1 or 3 (possibly both), so we need to evaluate $|A \cup B| = |A| + |B| - |A \cap B|$. To form A, we take the subsets of $\{2,3,4,5\}$ and put 1 in each, so $|A| = 2^4$. Similarly, $|B| = 2^4$. Now $A \cap B$ consists of those subsets of $\{1,2,3,4,5\}$ which contain both 1 and 3, and these are formed by putting 1 and 3 into each subset of $\{2,4,5\}$, so $|A \cap B| = 2^3$. Thus $|A \cup B| = 2^4 + 2^4 - 2^3 = 24$.

Finally, one could count all subsets of $\{1,2,3,4,5\}$, and subtract from that the number of subsets that contain neither 1 nor 3. This latter would be the number of subsets of $\{2,4,5\}$, so we obtain $2^5 - 2^3 = 24$.

The answer is A.

2. [1 mark] For exactly which of the predicates $P(n)$ shown below is the statement "For each $n \geq 0$, $P(n)$ implies $P(n+1)$" true?

(i) $\displaystyle\sum_{i=0}^{n} i = n^2 + n.$ (ii) $\displaystyle\sum_{i=0}^{n} i = \frac{n(n+1)}{2}.$ (iii) $\displaystyle\sum_{i=0}^{n} i = 2.$

A: (i)	B: (ii)	C: (ii), (iii)	D: (i), (iii)	E: All of them

Solution:

(i) The implication is not true for all n. For example, $P(0)$ is true, since $\displaystyle\sum_{i=0}^{0} i = 0 = 0^2 + 0$, but $P(1)$ is false, since $\displaystyle\sum_{i=0}^{1} i = 0 + 1 = 1 \neq 1^2 + 1$. Thus "If $P(0)$, then $P(1)$" is false.

(ii) The implication is true for all $n \geq 0$. This was proven in an early example on induction in the text.

(iii) The implication is true for all n, since $P(n)$ is false for all n. $P(0)$ is the assertion that $0 = 2$, which is false, $P(1)$ is the assertion that $1 = 2$, which is false, and for $n \geq 2$, $\displaystyle\sum_{i=0}^{n} i \geq 0 + 1 + 2 = 3 > 2$, so $P(n)$ is false for all $n \geq 2$.

Thus (ii) and (iii) make the universally quantified conditional statement true, while (i) does not. The answer is C.

3. [1 mark] In how many ways can five men and three women be arranged about a circular table if no two women are to sit side by side?

A: $8!/2$	B: $5!3!$	C: $8!$	D: $5!4!/2$	E: None of A, B, C, D

Solution: There are $(5-1)! = 4!$ ways to place 5 men around a circular table. Each such arrangement provides 5 spaces between consecutive men. We must choose 3 of these spaces and then arrange the 3 women in these 3 spaces. Since the seated men serve as markers, we may think of the problem of arranging the three women on the five remaining chairs as a permutation problem. Label the 5 available chairs as chairs 1 through 5, and label the three women as woman 1, woman 2 and woman 3. We now count the number of permutations of 5 chairs, taken 3 at a time, where the first chair will be assigned to woman 1, the second chair assigned to woman 2, and the third chair assigned to woman 3. Thus there are $P(5,3)$ ways to assign the 3 women to the 5 available chairs. As well, since the number of ways of arranging the 3 women was independent of how the 5 men had been arranged, the rule of product applies, and we find that there are $4!\,P(5,3) = 4!5!/2! = 5!4!/2$ such arrangements.

The answer is D.

4. [1 mark] If the sequence a_n, $n \geq 0$, is defined by: $a_0 = 2$, $a_1 = 5$, and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$, then a_4 is equal to:

A: 21	B: 11	C: 19	D: 13	E: None of A, B, C, D

Solution: We have $a_0 = 2$, $a_1 = 5$, $a_2 = 7$, $a_3 = 12$, and so $a_4 = 19$. The answer is C.

5. [1 mark] Suppose that $P(n)$ is a predicate and that $P(0)$ is true. The validity of exactly which of the following statements would allow us to conclude that $P(n)$ is true for all $n \in \mathbb{N}$?

   (i) For all $n \geq 0$, $P(n)$ implies $P(n+1)$.
   (ii) For all $n \geq 0$, $[\,P(0)$ and $P(1)$ and $\cdots$ and $P(n)\,]$ implies $P(n+1)$.
   (iii) For all $n \geq 1$, $[\,P(n-1)$ and $P(n)\,]$ implies $P(n+1)$.

A: (i), (iii)	B: (ii)	C: (ii), (iii)	D: (i)	E: None of A, B, C, D

Solution:
(i) True. This is the first principle of mathematical induction.
(ii) True, for this is exactly the statement of the second principle of mathematical induction.
(iii) False. For consider the predicate $P(n)$: n is even. $P(0)$ is true, and since $[\,P(n)$ and $P(n+1)\,]$ is false for every $n \geq 0$, it is true that $[\,P(n)$ and $P(n+1)\,]$ implies $P(n+2)$ for every integer $n \geq 0$, but $P(1)$ is false.

Thus (i) and (ii) are true, while (iii) is false. The answer is E.

6. [1 mark] Suppose that A, B and C are sets with $|A| = 11$, $|B| = 13$, $|C| = 14$, $|A \cap B| = 6$, $|A \cap C| = 6$, $|A \cap B \cap C| = 4$, and $|A \cup B \cup C| = 20$. What is $|B - C|$?

A: 3	B: 4	C: 6	D: 12	E: 13

Solution: Since $|A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)| = |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)| = |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$, we find that $20 = 11 + 13 + 14 - 6 - 6 - |B \cap C| + 4$, or $|B \cap C| = 10$. Thus $|B - C| = |B - (B \cap C)| = |B| - |B \cap C| = 13 - 10 = 3$. The answer is A.

7. [1 mark] Let $A = \{1, 2, 4, 5\}$, $B = \{1, 2, 3, 5, 7\}$ and $C = \{1, 3, 4, 8\}$. Exactly which of the following statements are true?

(i) $A \cup (B - C) = \{1, 2, 4, 5, 7\}$.
(ii) $A \cap (B \cup C) = \{1, 2, 3, 4, 5\}$.
(iii) $(B \cup A) - (B \cup C) = \varnothing$.
(iv) $(B - A) - C = \{7\}$.

A: (i), (iii)	B: (i), (iv)	C: (i), (ii), (iii)	D: (i), (iii), (iv)	E: (iii), (iv)

Solution:
(i) True, since $B - C = \{2, 5, 7\}$, and so $A \cup (B - C) = \{1, 2, 4, 5, 7\}$.
(ii) False, since $A \cap (B \cup C)$ is a subset of A, and A does not include 3, for example.
(iii) True, since $(B \cup A) - (B \cup C) = A - (B \cup C) = (A - B) - C = \{4\} - \{1, 3, 4, 8\} = \varnothing$.
(iv) True, since $(B - A) - C = \{3, 7\} - \{1, 3, 4, 8\} = \{7\}$.
Since only (ii) is false, the answer is D.

8. [1 mark] How many sequences of length 4 with entries from the set $\{1, 2, 3, 4, 5\}$ begin with either 1 or 2 (repetitions are permitted)?

A: $5! - 4!$	B: $2(5!)$	C: 5^4	D: $(2)(5)(4)(3)$	E: $2(5^3)$

Solution: We count those with first entry 1, then those with first entry 2. When the first entry is 1, we may have any element in the second position, any element in the third position, and any element in the fourth position. The rule of product applies, so there are 5^3 sequences that begin with 1. Similarly, there are 5^3 sequences that begin with 2, so there are altogether $2(5^3)$ sequences that begin either with 1 or 2. The answer is E.

9. [1 mark] Suppose that $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A - B = \{1, 3, 7\}$, and $B - A = \{2, 6, 8\}$. Then

A: $A = \{1, 3, 4, 5, 7\}$ and $B = \{6, 7, 8\}$;
B: $A = \{1, 3, 4, 7\}$ and $B = \{2, 4, 6, 8\}$;
C: $A = \{1, 2, 3, 6, 7, 8\}$ and $B = \{2, 4, 6, 8\}$;
D: $A = \{1, 3, 7\}$ and $B = \{2, 4, 6, 8\}$.
E: $A = \{1, 3, 4, 5, 7\}$ and $B = \{2, 4, 5, 6, 8\}$;

Solution: $A \cap B = (A \cup B) - (A \triangle B) = (A \cup B) - [(A - B) \cup (B - A)] = \{1, 2, 3, 4, 5, 6, 7, 8\} - \{1, 2, 3, 6, 7, 8\} = \{4, 5\}$. Thus $A = (A - B) \cup (A \cap B) = \{1, 3, 4, 5, 7\}$, and $B = (B - A) \cup (A \cap B) = \{2, 4, 5, 6, 8\}$. The answer is E.

10. [1 mark] How many sequences of length 3 with entries from the set $\{1, 2, 3, 4, 5\}$ have no repeated entries?

A: $5!$	B: 3^5	C: $(5)(4)(3)$	D: $3!$	E: 5^3

Solution: We have 5 choices for the first element, then 4 choices for the second element, and finally 3 choices for the third element. The rule of product applies, and so there are (5)(4)(3) such sequences. The answer is C.

11. [1 mark] If A and B are subsets of a set S, then $(A \cap B) \cup (A \cap B^c)$ is equal to:

A: $A \cup B^c$	B: $A \cap B^c$	C: B	D: A	E: None of A, B, C, D

Solution: We have $(A \cap B) \cup (A \cap B^c) = A \cap (B \cup B^c) = A \cap S = A$, so the answer is D.

12. [1 mark] Exactly which of the following statements are true?

(i) For all positive integers n, $(n^2)! = (n!)^2$.
(ii) For all positive integers n, $(n+1)! = (n+1)(n!)$.
(iii) For all positive integers n, $\dfrac{(n+2)!}{n!} = n^2 + 3n + 1$.

A: (i)	B: (ii)	C: (iii)	D: (i), (iii)	E: (ii), (iii)

Solution:
(i) False. For example, $2^2! = 4! = 24$, while $(2!)^2 = 2^2 = 4$.
(ii) True, by the inductive definition of factorials.
(iii) False. $(n+2)! = (n+2)(n+1)! = (n+2)(n+1)n!$, so $(n+2)!/n! = (n+2)(n+1) = n^2 + 3n + 2 \neq n^2 + 3n + 1$.
Since only (ii) is true, the answer is B.

13. [1 mark] Given a set S, exactly which of the following are true for all subsets A, B and C of S?

(i) $A \subseteq B$ if and only if $A \bigtriangleup B = \varnothing$.
(ii) $A \subseteq B$ if and only if $A \cup B = B$.
(iii) $A - B = A - C$ implies $B = C$.
(iv) $A \cap B = \varnothing$ and $A \cup B = S$ implies $A = B^c$.

A: All of them	B: (i), (iii), (iv)	C: (i), (iii)	D: (ii), (iv)	E: (ii), (iii), (iv)

Solution: (i) False, since $A \bigtriangleup B = \varnothing$ if and only if $A = B$. (ii) True, (result from text). (iii) False. Consider $A = \varnothing$. (iv) True, since $B^c = S - B = (A \cup B) - B = A - B = A - (A \cap B) = A - \varnothing = A$.
Since (ii) and (iv) are true, the answer is D.

14. [1 mark] For all sets A and B, $A \cup (A \bigtriangleup B)$ is equal to:

A: $A \cup B$	B: $A \cap B$	C: A	D: $A - B$	E: $B - A$

Solution: $A \cup (A \bigtriangleup B) = A \cup (A - B) \cup (B - A) = A \cup (B - A) = A \cup B$. The answer is A.

15. [1 mark] Let $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{1, 2, 3\}$. Exactly which of the following statements are true?

(i) $A \subseteq B$ (ii) $A \not\subseteq B$
(iii) $A \subseteq C$ (iv) $B \not\subseteq C$

A: (ii), (iii)	B: (ii)	C: (i), (iii)	D: (ii), (iv)	E: (ii), (iii), (iv)

Solution:

(i) False, since $2 \in A$ but $2 \notin B$.
(ii) True by (i).
(iii) True, for $1 \in C$ and $2 \in C$. Thus every element of A is also an element of C.
(iv) False, since $1 \in C$ and $3 \in C$, so $B \subseteq C$.
Since (ii) and (iii) are true, while (i) and (iv) are false, the answer is A.