Student's Name [print] — Student Number

Math 222a Second Midterm, Exam Code 111 November 9, 2002, 9:30am–noon

Instructions: Print your name and your instructor's name on the answer sheet. Sign the answer sheet, and mark your student number and section on the answer sheet. Use a PENCIL to mark your answers to questions 1–20 on the SCANTRON answer sheet, and as well, circle your answers on the question sheet.

1. Recall that for any positive integer n, $J_n = \{1, 2, \ldots, n\}$.
2. You may use the fact that $S(5,3) = 150$ at any time without justification.

2 marks A1. The number of symmetric relations on J_4 that contain $(1,2)$ is:

A: 2^6	B: 2^9	C: 2^{10}	D: 2^{11}	E: 2^{12}

Solution: A symmetric relation that contains $(1,2)$ must also contain $(2,1)$. Thus we may choose any of the $2^4 = 16$ subsets of the identity relation, together with $(1,2)$ and $(2,1)$, and then put with that any subset of the set of pairs (a,b) with $a < b$, making sure that for each such pair (a,b) that we choose, we also put (b,a) in as well. Since there are $(4^2-4)/2 = 6$ pairs (a,b) with $a < b$, one of which is $(1,2)$, there are $2^4 2^{6-1} = 2^9$ such relations on J_4. The answer is B.

2 marks A2. Exactly which of the relations R_1, R_2, and R_3 on J_4 that are given below are antisymmetric?

$$R_1 = \{(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (4,4), (4,1)\};$$
$$R_2 = \{(1,1), (1,2), (2,3), (3,3), (3,2), (4,2)\};$$
$$R_3 = \varnothing.$$

A: $R_1,\ R_3$	B: $R_2,\ R_3$	C: $R_1,\ R_2$	D: R_2	E: None of A, B, C, D

Solution:
R_1 is antisymmetric, since of the non-diagonal pairs in R_1, we do not find two of the form (a,b) and (b,a).
R_2 is not antisymmetric, since among the non-diagonal pairs in R_2, we find $(2,3)$ and $(3,2)$.
R_3 is antisymmetric, since there are no non-diagonal pairs at all (not any pairs in fact).
Thus R_1 and R_3 are antisymmetric, while R_2 is not antisymmetric. The answer is A.

2 marks A3. Exactly which of the relations R_1, R_2, R_3 on $\mathbb{Z}^+$ (the positive integers) shown below are reflexive?

$$R_1 = \{(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid a^2 - b^2 = 3(a-b)\};$$
$$R_2 = \{(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid a \le 2b\};$$
$$R_3 = \{(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid |a-b| < 4\}.$$

A: $R_1,\ R_3$	B: $R_2,\ R_3$	C: $R_1,\ R_2$	D: R_3	E: None of A, B, C, D

Solution:

R_1 is reflexive, since for any positive integer a, $a^2 - a^2 = 0 = 3(0) = 3(a - a)$, and so $(a, a) \in R_1$ for all $a \in \mathbb{Z}^+$.
R_2 is reflexive, since for any positive integer a, $a \leq 2a$, and so $(a, a) \in R_2$ for all $a \in \mathbb{Z}^+$.
R_3 is reflexive, since for any positive integer a, $|a - a| = 0 < 4$, and so $(a, a) \in R_3$ for all $a \in \mathbb{Z}^+$.
Thus R_1, R_2, and R_3 are reflexive. The answer is E.

2 marks A4. Exactly which of the relations R_1, R_2, R_3 on $\mathbb{Z}^+$ (the positive integers) shown below are antisymmetric?

$$R_1 = \{ (a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid a^2 - b^2 = 3(a - b) \};$$
$$R_2 = \{ (a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid a \leq 2b \};$$
$$R_3 = \{ (a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid |a - b| < 4 \}.$$

A: $R_1,\ R_3$	B: $R_2,\ R_3$	C: $R_1,\ R_2$	D: R_2	E: None of A, B, C, D

Solution:
R_1 is not antisymmetric, since there are positive integers a and b with $a \neq b$, yet $(a, b) \in R_1$ and $(b, a) \in R_1$. Since $a^2 - b^2 = (a - b)(a + b)$, we see that for $a^2 - b^2 = 3(a - b)$ with $a \neq b$, we must have $a + b = 3$. Thus $(1, 2) \in R_1$ and $(2, 1) \in R_1$, so R_1 is not antisymmetric.
R_2 is not antisymmetric, since it is possible to have positive integers a and b with $a \neq b$, yet $(a, b) \in R_2$ and $(b, a) \in R_2$. This would require that $a \leq 2b$ and $b \leq 2a$, which we may arrange by taking $a = 1$ and $b = 2$.
R_3 is not antisymmetric, since (for example) $|1 - 2| = |2 - 1| = 1 < 4$ so $(1, 2) \in R_3$ and $(2, 1) \in R_3$, but $1 \neq 2$.
Thus each of the three relations fails to be antisymmetric. The answer is E.

2 marks A5. Let R_1, R_2 and R_3 be the equivalence relations defined on J_5 by

$$R_1 = \{ (a, b) \in J_5 \times J_5 \mid a = b \text{ or } (a^2 \geq 5 \text{ and } b^2 \geq 5) \};$$
$$R_2 = \{ (a, b) \in J_5 \times J_5 \mid a^2 - 5a = b^2 - 5b \};$$
$$R_3 = \{ (a, b) \in J_5 \times J_5 \mid 5 \text{ is a factor of } a^2 - b^2 \}.$$

The partition $\{ \{ 1, 4 \}, \{ 2, 3 \}, \{ 5 \} \}$ of J_5 is the set of equivalence classes for exactly which of these equivalence relations?

A: R_1	B: R_2	C: R_3	D: $R_2,\ R_3$	E: R_1, R_2

Solution: The partition J_5/R_1 is equal to $\{ \{ 1 \}, \{ 2 \}, \{ 3, 4, 5 \} \}$, which is not the given partition.
Since $a^2 - 5a = b^2 - 5b$ if and only if $a^2 - b^2 = 5(a - b)$, equivalently $(a - b)(a + b) = 5(a - b)$, we see that $(a, b) \in R_2$ if and only if $a = b$ or else $a + b = 5$. Thus

$$R_2 = \{ (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 4), (4, 1), (2, 3), (3, 2) \},$$

and so the partition of A that is produced by the equivalence classes of R_2 is $\{ \{ 1, 4 \}, \{ 2, 3 \}, \{ 5 \} \}$.
Since 5 is a factor of $a^2 - b^2 = (a - b)(a + b)$ if and only if 5 is a factor of either $a - b$ or else $a + b$, we see that $R_3 = \{ (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 4), (4, 1), (2, 3), (3, 2) \} = R_2$, so the partition of J_5 that is produced by the equivalence classes of R_3 is also $\{ \{ 1, 4 \}, \{ 2, 3 \}, \{ 5 \} \}$.
Thus both R_2 and R_3 produce the desired partition of J_5, while R_1 does not. The answer is D.

2 marks A6. The number of equivalence relations on J_5 is:

A: 2^6	B: 15	C: 52	D: 125	E: 203

Solution: There is a natural bijective correspondence between the set of all equivalence relations on J_5 and the set of all partitions of J_5. Thus the number of equivalence relations on J_5 is equal to the number of partitions of J_5, which is B_5. Now $B_{n+1} = \sum_{i=0}^{n} \binom{n}{i} B_i$, and we have $B_0 = 1 = B_1$, $B_2 = 2$, $B_3 = \sum_{i=0}^{2} \binom{2}{i} B_i = 1 + 2 + 2 = 5$, $B_4 = \sum_{i=0}^{3} \binom{3}{i} B_i = 1 + 3 + 3(2) + 5 = 15$, and so $B_5 = \sum_{i=0}^{4} \binom{4}{i} B_i = 1 + 4 + 6(2) + 4(5) + 15 = 52$. The answer is C.

2 marks A7. The number of equivalence relations on J_5 with exactly 3 equivalence classes is:

A: 15	B: 75	C: 25	D: 90	E: None of A, B, C, D

Solution: This is the value of $s(5,3) = S(5,3)/3!$. Since $S(5,3) = 150$, the number is $150/6 = 25$. The answer is C.

2 marks A8. How many relations on J_{100} have the property of being both an equivalence relation and a partial order relation?

A: 0	B: 1	C: 100	D: 2^{100}	E: None of A, B, C, D

Solution: A relation on J_{100} is both an equivalence relation and a partial order relation if and only if it is reflexive, symmetric, antisymmetric, and transitive. By a result in the text, a relation is symmetric and antisymmetric if and only if it is a subset of the identity relation, while a relation is reflexive if and only if it contains the identity relation. Thus the only such relation is the identity relation on J_{100}, which is indeed transitive, hence is reflexive, symmetric, antisymmetric, and transitive.

The answer is B.

2 marks A9. Consider the relations R_1, R_2 and R_3 on $\mathbb{Z}^+$ (the positive integers) defined by

$$R_1 = \{\, (a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid a^2 - a = b^4 - b^2 \,\};$$
$$R_2 = \{\, (a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid |a-b| < 4 \,\};$$
$$R_3 = \{\, (a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid a+b \text{ is odd} \,\}.$$

Exactly which of the relations R_1, R_2, R_3 are equivalence relations?

A: R_1	B: R_3	C: $R_1,\ R_2$	D: $R_1,\ R_3$	E: None of them

Solution: Since $(2,2) \notin R_1$, R_1 is not reflexive and so R_1 is not an equivalence relation.

Since $(0,0) \notin R_3$, R_3 is not reflexive and is therefore not an equivalence relation.

R_2 is reflexive since $|a-a| = 0 < 4$ for every $a \in \mathbb{Z}^+$, and R_2 is symmetric, since $|a-b| < 4$ implies that $|b-a| = |a-b| < 4$. However, R_2 is not transitive, since $|0-3| = 3 < 4$ and $|3-6| = 3 < 4$, but $|0-6| = 6 \not< 4$. Thus R_2 is not an equivalence relation either.

The correct answer is E.

2 marks A10. The number of functions from J_3 to J_4 is:

A: 4^3	B: 3×4	C: 2^{12}	D: $3!$	E: 3^4

Solution: We want to determine the number of ways we can fill in the blanks in the construction $\{\, (1,\_), (2,\_), (3,\_) \,\}$ with elements of J_4. Since J_4 has 4 elements, there are 4 choices for each blank, so there are 4^3 ways to fill in the blanks. Thus there are 4^3 functions from J_3 to J_4. The answer is A.

2 marks A11. The number of injective functions from J_4 to J_3 is:

A: 0	B: 4!	C: 3!	D: 3×4	E: None of A, B, C, D

Solution: If there exists an injective function from a finite set A to a finite set B, then B has at least as many elements as does A. Since $|A| = 4 > 3 = |B|$, no function from A to B is injective. Thus the number of injective functions from J_4 to J_3 is 0. The answer is A.

2 marks A12. The number of surjective functions from J_4 to J_3 is:

A: 36	B: 24	C: 60	D: 0	E: None of A, B, C, D

Solution: This is $S(4,3) = \binom{4}{2}3! = 36$. The answer is A.

2 marks A13. How many left inverses does the function $f = \binom{1\,2\,3}{1\,4\,5}$ from J_3 to J_5 have?

A: 0	B: 1	C: 3	D: 5	E: 9

Solution: $3^{5-3} = 3^2 = 9$. The answer is E.

2 marks A14. Exactly which of the following functions from J_3 to J_4 are injective?

$$f = \{\,(1,3),\ (2,2),\ (3,3)\,\};$$
$$g = \{\,(1,4),\ (2,2),\ (3,3)\,\};$$
$$h = \{\,(1,1),\ (2,2),\ (3,3)\,\}.$$

A: f	B: g	C: h	D: $f,\ g$	E: $g,\ h$

Solution: f is not injective since $f(1) = 3 = f(3)$.
g is injective since $g(1) = 4$, $g(2) = 2$ and $g(3) = 2$, so distinct inputs to g result in distinct outputs.
h is injective since $h(1) = 1$, $h(2) = 2$ and $h(3) = 3$, so distinct inputs to h result in distinct outputs.
Thus both g and h are injective, while f is not. The answer is E.

2 marks A15. Which of the following properties are true for all finite sets A and all functions $f : A \to A$?

(i) If $(f \circ f)(x) = x$ for all $x \in A$, then f is a bijection.
(ii) If f is a bijection, then there is some $x \in A$ with $(f \circ f)(x) = x$.
(iii) If f is injective, then f has a right inverse.

A: (i), (ii)	B: (i), (iii)	C: (ii), (iii)	D: (i)	E: None of A, B, C, D

Solution:
(i) is true. For $f \circ f = 1_A$ implies that $f = f^{-1}$, so f is bijective.
(ii) is false. For consider $A = J_3$ and $f = \binom{1\,2\,3}{2\,3\,1}$, so that $f \circ f = \binom{1\,2\,3}{3\,1\,2}$.
(iii) is true. For f has a right inverse if and only if f is surjective, and since A is finite, f is surjective if and only if f is injective. Thus f has a right inverse if f is injective.
Thus (i) and (iii) are true, while (ii) is false. The answer is B.

2 marks A16. Which of the following statement are true for all sets A?

(i) If R is an equivalence relation on A, then $R \circ R$ is an equivalence relation on A.
(ii) If R is a partial order relation on A, then $R \circ R$ is a partial order relation on A.
(iii) If R is a relation on A but R is not an equivalence relation, then $R \circ R$ is not an equivalence relation on A.

A: (i)	B: (ii)	C: (iii)	D: (i), (ii)	E: (i), (iii)

Solution:
(i) is true, since $R \circ R = R$ for a reflexive transitive relation R.
(ii) is true, since $R \circ R = R$ for a reflexive transitive relation R.
(iii) is false. For consider $R = \{ (1,2), (2,1) \}$ on J_2. Since R is not reflexive, R is not an equivalence relation on J_2, but $R \circ R = \{ (1,1), (2,2) \} = 1_{J_2}$, which is an equivalence relation on J_2.
Thus (i) and (ii) are true, while (iii) is false. The answer is D.

2 marks A17. Let $A = J_{20} - \{ 1 \}$, so that A is a set of size 19, and let $R = \{ (a,b) \in A \times A \mid a \text{ divides } b \}$. You may assume that R is a partial order relation on A. The number of elements in A which are neither minimal elements nor maximal elements is:

A: 1	B: 6	C: 0	D: 5	E: 10

Solution: The primes between 2 and 20 are minimal elements, and each element x that is less then or equal to $20/2 = 10$ divides $2x$, and $2x$ is still in the set, while any element that is greater than 10 is not a divisor of any element of the set, so is maximal. Thus the non-maximal, non-minimal elements are the non-primes less than or equal to 10, namely 4,6,8,9,10. The answer is D.

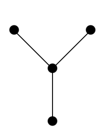
2 marks A18. Exactly which of the following relations are partial orders on the indicated sets?

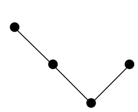
(i) The relation $R_1 = \{ (a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid a \geq b \}$ on the set $\mathbb{Z}^+$;
(ii) The relation $R_2 = \{ (a,b) \in J_6 \times J_6 \mid a^2 = b^2 \}$ on the set J_6;
(iii) The relation $R_3 = \{ (a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid |a-b| > 4 \}$ on the set $\mathbb{Z}^+$.

A: R_1	B: $R_2,\ R_3$	C: $R_1,\ R_2$	D: R_3	E: None of A, B, C, D

Solution: R_1 is a standard example of a partial order.
R_2 is the identity relation on J_6, since $a^2 = b^2$ implies that $(a-b)(a+b) = 0$ and if $a > 0$ and $b > 0$, $a + b \neq 0$ and so we must have $a - b = 0$; that is, $a = b$. Since the identity relation is reflexive, antisymmetric (as well as being symmetric), and transitive, R_2 is a partial order relation on J_6.
R_3 is not transitive.
The answer is C.

2 marks A19. Which of the following diagrams is the Hasse diagram of the partial order relation $R = \{ (1,1), (2,1), (3,1), (4,1), (2,2), (3,3), (2,4), (3,4), (4,4) \}$ on the set J_4?


(i)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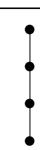
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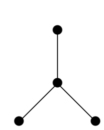
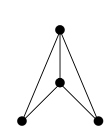

(iv)


(v)

A: (i)	B: (ii)	C: (iii)	D: (iv)	E: (v)

Solution: R has two minimal elements, 2 and 3, and a maximum element, namely 1, with 4 above both 2 and 3, so the correct diagram is (iv). The answer is D.

2 marks A20. Exactly which of the following relations are total orders on the indicated sets?

(i) The relation $R_1 = \{ (a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid a \geq b \}$ on the set $\mathbb{Z}^+$;
(ii) The relation $R_2 = \{ (a,b) \in J_6 \times J_6 \mid a^2 = b^2 \}$ on the set J_6;
(iii) The relation $R_3 = \{ (a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid |a-b| > 4 \}$ on the set $\mathbb{Z}^+$.

A: R_1	B: $R_2,\ R_3$	C: $R_1,\ R_2$	D: R_3	E: None of A, B, C, D

Solution: R_1 is a standard example of a total order.
R_2 is the identity relation on J_6, which is not a total order. For example, neither $(1,2)$ nor $(2,1)$ is in R_2.
R_3 is not a partial order, so it is not a total order.
Thus only R_1 is a total order. The answer is A.

B1. In each case, either give an example of one relation R on the set J_3 with all of the indicated properties or explain why no such relation exists.

2 marks (a) reflexive and symmetric, but neither antisymmetric nor transitive.

Solution: Consider $R = \{ (1,1), (2,2), (3,3), (1,2), (2,1), (2,3), (3,2) \}$. Since $1_{J_3} \subseteq R$, R is reflexive. Since $R^{-1} = \{ (1,1), (2,2), (3,3), (2,1), (1,2), (3,2), (2,3) \} = R$, R is symmetric. Since $R \cap R^{-1} = R$, and R is not a subset of 1_{J_3}, it follows that R is not antisymmetric. Finally, $(1,2) \in R$ and $(2,3) \in R$, but $(1,3) \notin R$, so R is not transitive.

2 marks (b) antisymmetric and transitive, but neither reflexive nor symmetric.

Solution: $R = \{ (1,1), (2,2), (1,2), (2,3), (1,3) \}$. Since $(3,3) \notin R$, R is not reflexive. Since $(1,2) \in R$ but $(2,1) \notin R$, R is not symmetric. Since $R^{-1} = \{ (1,1), (2,2), (2,1), (3,2), 3,1) \}$, we see that $R \cap R^{-1} = \{ (1,1), (2,2) \} \subseteq 1_{J_3}$, so R is antisymmetric. Finally,

$$R \circ R = \{ (1,1), (2,2), (1,2), (2,3), (1,3) \} \subseteq R$$

(actually, they are equal), so R is transitive.

B2. Let R be the relation from J_6 to J_4 that is given by

$$R = \{ (1,1), (2,1), (4,3), (4,4), (5,4), (6,4) \},$$

and let S be the relation from J_4 to J_3 that is given by

$$S = \{ (1,1), (1,3), (2,2), (3,3), (4,2) \}.$$

1 mark (a) What is the image of R?

Solution: The image of R is $\{ 1, 3, 4 \}$.

2 marks (b) Calculate $S \circ R$.

Solution: $S \circ R = \{ (1,1), (1,3), (2,1), (2,3), (4,3), (4,2), (5,2), (6,2) \}$.

B3. In each case, either prove that the statement is true, or else provide a counterexample.

3 marks (a) For any set A and any relations R and S on A, if R is reflexive, then $R \cap S$ is reflexive.

Solution: This is not necessarily true. Let $A = \{ 1 \}$, $R = \{ (1,1) \}$ and $S = \emptyset$. Then R is reflexive, but $R \cap S = \emptyset$, which is not reflexive.

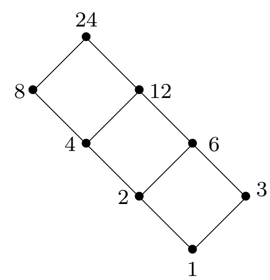
3 marks (b) For any set A and any relations R and S on A, if R and S are transitive, then $R \cap S$ is transitive.

Solution: Let R and S be transitive relations on A, and let $(a,b),(b,c) \in R \cap S$. Then since $(a,b),(b,c) \in R$ and R is transitive, we have $(a,c) \in R$. As well, $(a,b),(b,c) \in S$ and S is transitive, so $(a,c) \in S$. Thus $(a,c) \in R \cap S$. This proves that $R \cap S$ is transitive.

2 marks B4. Recall that $D(24)$ is the set of all positive divisors of 24. In this question, consider the partial order on this set given by $R = \{ (a,b) \in D(24) \times D(24) \mid a \text{ divides } b \}$.

Draw the Hasse diagram for this partially ordered set.

Solution: $D(24) = \{ 1, 2, 3, 4, 6, 8, 12, 24 \}$.



B5. Let $A = J_{10}$ and $R = \{ (x,y) \in A \times A \mid x^2 - y^2 \text{ is a multiple of } 6 \}$.

4 marks (a) Prove that R is an equivalence relation on A.

Solution: We prove that R is reflexive, symmetric, and transitive. Let $a \in J_{10}$. Since $a^2 - a^2 = 0 = (0)(6)$, $(a,a) \in R$, so R is reflexive. Suppose now that $a, b \in J_{10}$ are such that $(a,b) \in J_{10}$. Then $a^2 - b^2$ is a multiple of 6, so $b^2 - a^2 = -(a^2 - b^2)$ is a multiple of 6. Thus $(b,a) \in R$, and so R is symmetric. Finally, suppose that $a,b,c \in J_{10}$ are such that $(a,b),(b,c) \in R$. Then $a^2 - b^2$ and $b^2 - c^2$ are each multiples of 6, so $a^2 - c^2 = (a^2 - b^2) + (b^2 - c^2)$ is the sum of multiples of 6, hence is a multiple of 6. Thus $(a,c) \in R$, so R is transitive.

Since R is reflexive, symmetric, and transitive, R is an equivalence relation on J_{10}.

2 marks (b) Find $[5]_R$, the equivalence class of 5 under the equivalence relation R.

Solution: $[5]_R = \{ a \in J_{10} \mid a^2 - 25 \text{ is a multiple of } 6 \} = \{ 1, 5, 7 \}$. To verify this, we must work out $1^2 - 25 = -24 = (-4)(6)$, $2^2 - 25 = -21$, which is not a multiple of 6, $3^2 - 25 = -16$, which is not a multiple of 6, $4^2 - 25 = -9$, which is not a multiple of 6, $5^2 - 25 = 0 = (0)(6)$, $6^2 - 25 = -11$, which is not a multiple of 6, $7^2 - 25 = 24 = (4)(6)$, $8^2 - 25 = 39$, which is not a multiple of 6, $9^2 - 25 = 56$, which is not a multiple of 6, and $10^2 - 25 = 75$, which is not a multiple of 6.

3 marks B6. Either give an example of a set A and a surjective function $f : A \to J_6$ for which

$$|A/\mathrm{Ker}(f)| = 4$$

or else prove that this is impossible.

Solution: If $f : A \to J_6$ is surjective, then $f^{-1}(\{ i \}) \neq \varnothing$ for each $i \in J_6$. But $A/\mathrm{Ker}(f)$ has 6 cells, namely $A/\mathrm{Ker}(f) = \{ f^{-1}(\{1\}), f^{-1}(\{2\}), f^{-1}(\{3\}), f^{-1}(\{4\}), f^{-1}(\{5\}), f^{-1}(\{6\}) \}$. Thus this is impossible.

2 marks B7. $P = \{\{1, 2\}, \{3, 4\}\}$ is a partition of J_4. Give the equivalence relation R on J_4 that is determined by the partition P (write R as a set of ordered pairs).

Solution:

$$\begin{aligned} R &= (\{1, 2\} \times \{1, 2\}) \cup (\{3, 4\} \times \{3, 4\}) \\ &= \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4)\}. \end{aligned}$$

B8. In each case below, sets A and B and a function $f : A \to B$ are given. In each case, you must either find functions $g : A \to A$ and $h : A \to A$ such that $g \neq h$ and $f \circ g = f \circ h$, or else explain why no such functions exist.

3 marks (a) $A = J_3$, $B = J_4$ and $f : A \to B$ is $f = \binom{1\,2\,3}{1\,4\,3}$.

Solution: $f : J_3 \to J_4$ is injective, so f has a left inverse k say, so that $k \circ f = 1_{J_3}$. Suppose g and h are functions from J_4 to J_3 such that $f \circ g = f \circ h$. Then $k \circ (f \circ g) = k \circ (f \circ h)$, so $(k \circ f) \circ g = (k \circ f) \circ h$. Since $k \circ f = 1_{J_3}$, we have $g = 1_{J_3} \circ g = 1_{J_3} \circ h = h$. Thus there are no functions $g : A \to A$ and $h : A \to A$ such that $g \neq h$ and $f \circ g = f \circ h$.

3 marks (b) $A = J_4$, $B = J_3$ and $f : A \to B$ is $f = \binom{1\,2\,3\,4}{1\,2\,3\,1}$.

Solution: Since g is surjective, it has a right inverse. In fact, since $|f^{-1}(1)| = 2$ while all other preimage sets are singletons, there are exactly two right inverses, call them g and h. Then $g, h : J_3 \to J_4$ and $f \circ g = 1_{J_3} = f \circ j$, but $g \neq h$.

4 marks B9. How many functions from J_{10} to J_{15} are strictly increasing (a function $f : J_{10} \to J_{15}$ is said to be strictly increasing if for any $m, n \in J_{10}$ with $m < n$, $f(m) < f(n)$). Explain your answer.

Solution: To build such a function, we assign values to $b_1, \dots, b_{10}$ in the function prototype

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & b_7 & b_8 & b_9 & b_{10} \end{pmatrix}$$

in such a way that $1 \leq b_1 < b_2 < \cdots < b_{10} \leq 15$. Since for each choice of 10 elements from J_{15}, there is exactly one way to arrange them in ascending order, the number of ways to build such functions is equal to the number of subsets of J_{15} of size 10, $\binom{15}{10}$.

4 marks B10. Let A and B be sets, let S be a partial order relation on B, and let $f : A \to B$ be an injective function. Define a relation R on A by

$$R = \{(x, y) \in A \times A \mid (f(x), f(y)) \in S\}.$$

Prove that R is a partial order on A.

Solution: Let $a \in A$. Since $f(a) \in B$ and S is a reflexive relation on B, we have $(f(a), f(a)) \in S$, so $(a, a) \in R$. Thus R is reflexive.

Now suppose that $a, b \in A$ are such that $(a, b) \in R$ and $(b, a) \in R$. Then $(f(a), f(b)) \in S$, and $(f(b), f(a)) \in S$. Since S is antisymmetric, we must have $f(a) = f(b)$, and then since f is injective, we obtain that $a = b$. Thus R is antisymmetric.

Finally, suppose that $a, b, c \in A$ are such that $(a, b) \in R$ and $(b, c) \in R$. Then $(f(a), f(b)) \in S$ and $(f(b), f(c)) \in S$. Since S is transitive, we have $(f(a), f(c)) \in S$ and so by definition, $(a, c) \in R$. Thus R is transitive.

Since R is reflexive, antisymmetric and transitive, R is a partial order relation on A.

Bonus: Suppose that R is any reflexive and transitive relation on a set A. Let E be the relation on A given by

$$E = \{(x, y) \in A \times A \mid (x, y) \in R \text{ and } (y, x) \in R\}.$$

3 marks (a) Prove that E is an equivalence relation on A.

Solution: Let $a \in A$. Then since R is reflexive, $(a, a) \in R$, so $(a, a) \in E$. Thus E is reflexive.

Now suppose that $(a, b) \in E$. Then $(a, b) \in R$ and $(b, a) \in R$, so we have $(b, a) \in R$ and $(a, b) \in R$. Thus $(b, a) \in E$, and so E is symmetric.

Finally, suppose that $(a, b), (b, c) \in E$. Then $(a, b) \in R$, $(b, a) \in R$, $(b, c) \in R$, and $(c, b) \in R$. Since R is transitive, $(a, b) \in R$ and $(b, c) \in R$ implies that $(a, c) \in R$, while $(c, b) \in R$ and $(b, a) \in R$ implies that $(c, a) \in R$. Now, since $(a, c) \in R$ and $(c, a) \in R$, we have $(a, c) \in E$, as required. Thus E is transitive.

Since E is reflexive, symmetric, and transitive, E is an equivalence relation.

3 marks (b) For this part, you may assume that the relation E defined above is an equivalence relation on A. Now let R' be the relation on the quotient set A/E defined by

$$R' = \{ ([x]_E,\ [y]_E) \mid \text{there are } u \in [x]_E \text{ and } v \in [y]_E \text{ with } (u, v) \in R \}.$$

Prove that R' is a partial order relation on A/E.

Solution: Observe that

$$R' = \{ ([x]_E,\ [y]_E) \mid (x, y) \in R \}.$$

For if $([x]_E, [y]_E) \in R'$, then there exists $u \in [x]_E$ and $v \in [y]_E$ with $(u, v) \in R$. Now $u \in [x]_E$ implies that $(x, u) \in E$, and so $(x, u) \in R$. Similarly, $v \in [y]_E$ implies that $(v, y) \in R$. Thus $(x, u) \in R$, $(u, v) \in R$, and $(v, y) \in R$. By the transitivity of R, we have $(x, y) \in R$. On the other hand, if $(x, y) \in R$, then since $x \in [x]_E$ and $y \in [y]_E$, it follows that $([x]_E, [y]_E) \in R'$.

We now prove that R' is reflexive, antisymmetric, and transitive.

Let $[x]_E \in A/E$. Since R is reflexive, $(x, x) \in R$, so by the preceding remarks, $([x]_E, [x]_E) \in R'$. Thus R' is reflexive.

Now suppose that $([x]_E, [y]_E) \in R'$ and $([y]_E, [x]_E) \in R'$. Then $(x, y) \in R$ and $(y, x) \in R$, and so $(x, y) \in E$. Thus $[x]_E = [y]_E$, which proves that R' is antisymmetric.

Finally, suppose that $([x]_E, [y]_E) \in R'$ and $([y]_E, [z]_E) \in R'$. Then $(x, y) \in R$ and $(y, z) \in R$. Since R is transitive, we have $(x, z) \in R$, and so $([x]_E, [z]_E) \in R'$. Thus R' is transitive.

Since R' is reflexive, antisymmetric, and transitive, R' is a partial order relation.