Student's Name [print] | Student Number

Mathematics 222a Quiz 2 **CODE 222** **November 21, 2002**

Instructions: Print your name and student number at the top of this question sheet. Print your name and your instructor's name on the answer sheet. Sign the answer sheet, and mark your student number and section on the answer sheet. The **code** for this question sheet is shown above. Mark this number in the "code" box on the answer sheet. *Both sheets must be handed in at the end of the quiz!* Mark your answers to all questions (1–15) in the left column of the answer sheet.

Calculators are not permitted.

1. [1 mark] What is the **product** of 2 and 5 in $\mathbb{Z}_6$?

A: 0	B: 2	C: 3	D: 4	E: 5

Solution: $(2)(5) = 10 = 1(6) + 4$, so the product of 2 and 5 in $\mathbb{Z}_6$ is 4.
The answer is D.

2. [1 mark] What is the **multiplicative inverse** of 2 in $\mathbb{Z}_7$?

A: 0	B: 3	C: 4	D: 2	E: 8

Solution: $2^2 = 4$, and in $\mathbb{Z}$, $2^3 = 8 = 1(7) + 1$, so in $\mathbb{Z}_7$, $2^3 = 1$. Thus $7^{-1} = 2^2 = 4$.
The answer is C.

3. [1 mark] A binary operation $*$ is defined on $\mathbb{Z}$ by $x * y = x + y + 3$ for all x, y in $\mathbb{Z}$. You are told that $*$ has an identity. The **identity** of $*$ is:

A: -5	B: -3	C: 0	D: 3	E: 5

Solution: Let e denote the identity of $*$. Then for all $x \in \mathbb{Z}$, $x = e * x = e + x + 3$, so $e + 3 = 0$. Thus $e = -3$.
The answer is B.

4. [1 mark] In $\mathbb{Z}_5$ with operation **multiplication**, what is the number of invertible elements?

A: 5	B: 4	C: 3	D: 2	E: 1

Solution: The multiplication table for $\mathbb{Z}_5$ is:

	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Thus only 0 is not invertible. The answer is B.

5. [1 mark] In S_4, the group of all bijective functions from J_4 to J_4 under composition of functions, if $a = \binom{1\,2\,3\,4}{2\,4\,1\,3}$, then a^3 is equal to:

A: $\binom{1\,2\,3\,4}{4\,3\,2\,1}$	B: $\binom{1\,2\,3\,4}{2\,4\,3\,1}$	C: $\binom{1\,2\,3\,4}{2\,4\,1\,3}$	D: $\binom{1\,2\,3\,4}{3\,1\,4\,2}$	E: $\binom{1\,2\,3\,4}{1\,2\,3\,4}$

Solution: Since $a^2 = \binom{1\,2\,3\,4}{2\,4\,1\,3} \circ \binom{1\,2\,3\,4}{2\,4\,1\,3} = \binom{1\,2\,3\,4}{4\,3\,2\,1}$, we have $a^3 = a^2 \circ a = \binom{1\,2\,3\,4}{4\,3\,2\,1} \circ \binom{1\,2\,3\,4}{2\,4\,1\,3} = \binom{1\,2\,3\,4}{3\,1\,4\,2}$.
The answer is D.

6. [1 mark] Which of the following statements are true for every positive integer n?

(i) there is an abelian group of order n;
(ii) in every group of order n, there is some element with order n;
(iii) if G is a finite group and $g \in G$ has order n, then $g^{-1} = g^{n-1}$.

A: (i)	B: (ii)	C: (iii)	D: (i), (iii)	E: (i), (ii)

Solution:
(i) is true, since $\mathbb{Z}_n$ with addition is an abelian group of order n.
(ii) is false. For example, the direct product of two cyclic groups of order 2 is a group of order 4 with no element of order 4.
(iii) is true, for $g^n = e$, so $g^{n-1}g = gg^{n-1} = g^n = e$.
Since (i) and (iii) are true, while (ii) is false, the answer is D.

7. [1 mark] Which of the following binary operations on $\mathbb{Z}$ have identity elements?

(i) $x * y = (x+2)(y+2) - 2$;
(ii) $x * y = x - y$;
(iii) $x * y = xy$.

A: (i), (ii)	B: (i), (iii)	C: (i)	D: (ii)	E: (iii)

Solution:
(i) has an identity element. To determine what it is, let e denote the identity. Then $x = e * x = (e+2)(x+2) - 2$ and so $x(e+1) + 2(e+1) = 0$; that is, $(x+2)(e+1) = 0$ for every $x \in \mathbb{Z}$. Thus we must have $e = -1$. Since the operation is commutative, we have $x * (-1) = (-1) * x = (-1+2)(x+2) - 2 = x$ for every $x \in \mathbb{Z}$. Thus $e = -1$ is in fact an identity for the binary operation.
(ii) has no identity element. For if $e \in \mathbb{Z}$ was an identity for this operation, then $e = e * e = e - e = 0$. But then we would have $1 = 0 * 1 = 0 - 1 = -1$, which is not the case.
(iii) has 1 as its identity element.
Since (i) and (iii) each have identities, but (ii) does not, the answer is B.

8. [1 mark] Let $*$ denote the binary operation defined on $\mathbb{Q}$, the set of all rational numbers, by $x * y = xy + 1$ for all x, y in $\mathbb{Q}$. Exactly which of the following properties does $*$ have?

(i) commutativity;
(ii) associativity;
(iii) has an identity.

A: (i)	B: (i), (ii)	C: (i), (iii)	D: (ii), (iii)	E: (i), (ii), (iii)

Solution: The operation is commutative since $x * y = xy + 1 = yx + 1 = y * x$ for all $x, y \in \mathbb{Q}$.

The operation is not associative, since $x * (y * z) = x * (yz + 1) = xyz + x + 1$, while $(x * y) * z = (xy + 1) * z = xyz + z + 1$ for any $x, y, z \in \mathbb{Q}$. Thus for $x = 1, y = 0 = z$, we have $1 * (0 * 0) = 1 * 1 = 2$ while $(1 * 0) * 0 = 1 * 0 = 1$, so $1 * (0 * 0) \neq (1 * 0) * 0$.

Finally, $*$ does not have an identity. For if e was an identity for $*$, then for any $x \in \mathbb{Q}$, $x = x*e = xe+1$. In particular, from $x = 0$ we could conclude that $0 = 1$. Since $0 \neq 1$, it follows that there is no identity element for $*$.

Thus the binary operation has only property (i). The answer is A.

9. [1 mark] How many binary operations on J_5 have an identity element?

A: 5^{17}	B: 5^{25}	C: 5^{15}	D: 5^{16}	E: None of A, B, C, D

Solution: The table for any operation on J_5 that had 1 as an identity would look like:

	1	2	3	4	5
1	1	2	3	4	5
2	2	?	?	?	?
3	3	?	?	?	?
4	4	?	?	?	?
5	5	?	?	?	?

The 16 question marks could each be replaced by any of the 5 elements from J_5, so there are 5^{16} such operations on J_5. Similarly, there would be 5^{16} binary operations that had 2 as identity, and there would be 5^{16} binary operations that had 3 as identity. All told, there are $5(5^{16}) = 5^{17}$ binary operations on J_5 that have an identity.

The answer is A.

10. [1 mark] How many commutative binary operations are there on J_5?

A: 5^{17}	B: 5^{25}	C: 5^{15}	D: 5^{16}	E: None of A, B, C, D

Solution: If we consider how to construct an operation table so that the resulting operation is commutative, we see that the table must be symmetric about the main diagonal. Thus we have the 5 diagonal locations with 5 choices each, then each of the $4 + 3 + 2 + 1 = 10$ above diagonal locations with 5 choices each. The remainder of the table is now completely determined by the symmetry requirement. Thus there are $5^5 5^{10} = 5^{15}$ commutative binary operations on J_5.

The answer is C.

11. [1 mark] Recall that S_3 is the group of all bijective functions from J_3 to J_3 with binary operation being composition of functions. Exactly which of the following is true?

(i) S_3 is abelian;
(ii) S_3 is not cyclic;
(iii) S_3 contains an element of order 3.

A: (i)	B: (ii)	C: (iii)	D: (i), (ii)	E: None of A, B, C, D

Solution:

(i) is false. For example, $\binom{1\,2\,3}{1\,3\,2} \circ \binom{1\,2\,3}{2\,1\,3} = \binom{1\,2\,3}{3\,1\,2}$, while $\binom{1\,2\,3}{2\,1\,3} \circ \binom{1\,2\,3}{1\,3\,2} = \binom{1\,2\,3}{2\,3\,1}$.

(ii) is true. For every cyclic group is abelian, and as shown in (i), S_3 is not abelian. Thus S_3 is not cyclic.

(iii) is true. $\binom{1\,2\,3}{2\,3\,1} \circ \binom{1\,2\,3}{2\,3\,1} = \binom{1\,2\,3}{3\,1\,2}$, and $\binom{1\,2\,3}{3\,1\,2} \circ \binom{1\,2\,3}{2\,3\,1} = \binom{1\,2\,3}{1\,2\,3}$. Thus $\binom{1\,2\,3}{2\,3\,1}$ is an element of order 3 in S_3.

Since (i) is false, while (ii) and (iii) are true, the answer is E.

12. [1 mark] In the group of units U_{10} of $\mathbb{Z}_{10}$ under multiplication, the order of 9 is:

A: 5	B: 4	C: 3 :	D: 2	E: None of A, B, C, D

Solution: The order of 9 is the smallest positive exponent r such that $9^r = 1$ in $\mathbb{Z}_{10}$. In $\mathbb{Z}$, we have $9^2 = 81 = (8)(10) + 1$, so that in $\mathbb{Z}_{10}$, $9^2 = 1$. Thus 9 is invertible in $\mathbb{Z}_{10}$ under multiplication, and the order of 9 in U_{10} is 2.

The answer is D.

13. [1 mark] The operation table for the group D_4 is:

$\circ$	e	r	r^2	r^3	h	v	d^-	d^+
e	e	r	r^2	r^3	h	v	d^-	d^+
r	r	r^2	r^3	e	d^-	d^+	v	h
r^2	r^2	r^3	e	r	v	h	d^+	d^-
r^3	r^3	e	r	r^2	d^+	d^-	h	v
h	h	d^+	v	d^-	e	r^2	r^3	r
v	v	d^-	h	d^+	r^2	e	r	r^3
d^-	d^-	h	d^+	v	r	r^3	e	r^2
d^+	d^+	v	d^-	h	r^3	r	r^2	e

In D_4, the product $v^{-1}r^2$ is equal to:

A: h	B: d^-	C: r^3	D: v	E: e

Solution: Since $v^2 = e$, $v^{-1} = v$. Thus $v^{-1}r^2 = vr^2 = h$. The answer is A.

14. [1 mark] The operation table for the group D_4 is:

$\circ$	e	r	r^2	r^3	h	v	d^-	d^+
e	e	r	r^2	r^3	h	v	d^-	d^+
r	r	r^2	r^3	e	d^-	d^+	v	h
r^2	r^2	r^3	e	r	v	h	d^+	d^-
r^3	r^3	e	r	r^2	d^+	d^-	h	v
h	h	d^+	v	d^-	e	r^2	r^3	r
v	v	d^-	h	d^+	r^2	e	r	r^3
d^-	d^-	h	d^+	v	r	r^3	e	r^2
d^+	d^+	v	d^-	h	r^3	r	r^2	e

Exactly which of the following subsets of D_4 are subgroups of D_4?

(i) $\{e, d^-, d^+, v\}$;
(ii) $\{e, h, v, d^+\}$;
(iii) $\{r, r^2, r^3, e\}$.

A: (i)	B: (ii)	C: (iii)	D: (i), (iii)	E: (i), (ii)

Solution: Since D_4 is finite, a subset H of D_4 is a subgroup of D_4 if and only if H contains the identity and is closed under the binary operation.

(i) is not closed under the binary operation. For example, $d^-d^+ = r^2$, which is not in the set. Thus (i) is not a subgroup of D_4.

(ii) is not closed with respect to the binary operation. For example, $hv = r^2$, which is not in the set. Thus (ii) is not a subgroup of D_4.

(iii) is a subgroup of D_4; in fact, it is $\langle r \rangle$.

Thus only (iii) is a subgroup of D_4. The answer is C.

15. [1 mark] A binary operation $*$ is defined on $\mathbb{Z}$ by $x * y = xy - 2x - 2y + c$ for all x, y in $\mathbb{Z}$, where $c \in \mathbb{Z}$ is a constant. You are told that $*$ is associative. What is the value of c?

A: 6	B: 3	C: 0	D: -3	E: -6

Solution: $(x * y) * z = (xy - 2x - 2y + c) * z = xyz - 2xz - 2yz + cz - 2xy + 4x + 4y - 2c - 2z + c = xyz - 2(xy + yz + zx) + 4x + 4y + (c - 2)z - c$, while $x * (y * z) = x * (yz - 2y - 2z + c) = xyz - 2xy - 2xz + cx - 2x - 2yz + 4y + 4z - 2c + c = xyz - 2(xy + yz + zx) + (c - 2)x + 4y + 4z - c$, so for associativity we want $c - 2 = 4$. Thus $c = 6$.

Of course, since we are told that $*$ is associative, we may just carry out the above computation for specific values of x, y, and z to obtain an equation in which c is the only unknown. Try $x = y = z = 0$. We have $0 * (0 * 0) = 0 * (0 - 0 - 0 + c) = 0 - 0 - 2(c) + c = -c$, and $(0 * 0) * 0 = (0 - 0 - 0 + c) * 0 = 0 - 2c - 0 + c = -c$, so we learn only that $-c = -c$. Ok, try $x = y = 0$ and $z = 1$. We have $0 * (0 * 1) = 0 * (0 - 0 - 2 + c) = 0 - 0 + 4 - 2c + c = 4 - c$, while $(0 * 0) * 1 = (0 - 0 - 0 + c) * 1 = c - 2c - 2 + c = -2$, so $4 - c = -2$ and thus $c = 6$, as we found above.

The answer is A.