1. (a) State Rolle's Theorem.

   **Solution**:
   See either (i) Theorem 1.11 in Professor Shafikov's on-line notes or (ii) page 284 of Stewart. Or if you are in Professor Metzler's class, you can find the theorem in the lecture notes from January $10^{th}$.

   (b) State the Mean Value Theorem.

   **Solution**:
   See either (i) Theorem 1.12 in Professor Shafikov's on-line notes or (ii) page 285 of Stewart. Or if you are in Professor Metzler's class, you can find the theorem in the lecture notes from January $11^{th}$.

   (c) Use Rolle's Theorem to prove the Mean Value Theorem.

   **Solution**:
   See either (i) the proof of Theorem 1.12 in Professor Shafikov's on-line notes or (ii) page 286 of Stewart. Or if you are in Professor Metzler's class, you can find the proof in the lecture notes from January $12^{th}$.

2. Suppose $f$ is continuous on $[0, 2]$, differentiable on $(0, 2)$ and satisfies $f(0) = 0$, $f(2) = 2$. Prove that there exists a point $x \in (0, 2)$ such that $f'(x) = \dfrac{1}{f(x)}$.
   *Hint:* Consider the function $g(x) = [f(x)]^2$.

**Solution:**
As the product of continuous functions, $g$ is itself continuous on $[0,2]$. Similarly, as the product of differentiable functions, $g$ is itself differentiable on $(0,2)$. Thus $g$ satisfies the conditions of the Mean Value Theorem, which ensures that a point $x \in (0,2)$ exists with the property that

$$g'(x) = \frac{g(2) - g(0)}{2 - 0} .$$

Now $g'(x) = 2f(x)f'(x)$, $g(2) = 4$ and $g(0) = 0$, so that the above equality can be re-written as $2f(x)f'(x) = 2$, or $f'(x) = \frac{1}{f(x)}$.

3. (a) Evaluate $\displaystyle\int \ln x \, dx$.

   **Solution**:
   This is Example 2 in Section 7.1 of Stewart. Use integration by parts with $u = \ln x$ and $dv = dx$. This leads to $du = \frac{1}{x}dx$ and $v = x$, yielding

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} \, dx = x \ln x - \int 1 \, dx = x \ln x - x + C .$$

   (b) Evaluate $\displaystyle\int \frac{8x-3}{x^2-x} \, dx$.

   **Solution**:
   The denominator factors as $x^2 - x = x(x-1)$, leading to

$$\frac{8x-3}{x^2-x} = \frac{A}{x} + \frac{B}{x-1} \implies 8x - 3 = (A+B)x - A .$$

   Thus $A = 3$ and $B = 5$, and the integral is

$$\int \frac{8x-3}{x^2-x} \, dx = \int \frac{3}{x} \, dx + \int \frac{5}{x-1} \, dx = 3 \ln |x| + 5 \ln |x-1| + C .$$

4. Evaluate $\int e^{2x} \cos x \, dx$.

   **Solution**:

   This is a minor variation (and in fact easier version) of Problem 17, Section 7.1, which was a suggested exercise. The problem requires integration by parts twice (there are at least two ways to perform the integration). To this end let $I = \int e^{2x} \cos x \, dx$ and use integration by parts with $u = e^{2x}$ and $dv = \cos x \, dx$ (so that $du = 2e^{2x} \, dx$ and $v = \sin x$) to get

$$I = e^{2x} \sin x - 2 \int e^{2x} \sin x \, dx \ .$$

   Now use integration by parts again with $u = e^{2x}$ and $dv = \sin x \, dx$ (so that $du = 2e^{2x} \, dx$ and $v = -\cos x$) to get

$$I = e^{2x} \sin x - 2 \left[ -e^{2x} \cos x + 2 \int e^{2x} \cos x \, dx \right] = e^{2x} (\sin x + 2 \cos x) - 4I \ .$$

   Now solve for $I$ (and introduce a constant of integration) to find

$$I = \frac{1}{5} e^{2x} (\sin x + 2 \cos x) + C \ .$$

5. Find the partial fraction decomposition of $\dfrac{8x^3 + 19x^2 + 10x + 5}{(x^2 + 2x + 1)(x^2 + 1)}$. Form alone is not sufficient (that is, make sure you determine the numerical values of all coefficents).

   **Solution**

   The denomintor is not fully factored, since $x^2 + 2x + 1 = (x + 1)^2$. And since $x^2+1$ is irreducible, the complete factorization of the denominator is $(x^2+2x+1)(x^2+1) = (x+1)^2(x^2+1)$. We have one repeated linear factor and one irreducible quadratic, therefore the form of the decomposition is

$$\frac{8x^3+19x^2+10x+5}{(x^2+2x+1)(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}\,.$$

Multiplying through by $(x+1)^2(x^2+1)$ we get

$$\begin{aligned} 8x^3+19x^2+10x+5 &= A(x+1)\left(x^2+1\right) + B\left(x^2+1\right) + (Cx+D)(x+1)^2 \\ \\ &= (A+C)\,x^3 + (A+B+2C+D)\,x^2 \\ &\quad + (A+C+2D)\,x + (A+B+D) \end{aligned}$$

This system is easily solved: for example if $A+C=8$ and $A+C+2D=10$, then $D=1$. And if $A+B+2C+D=19$ and $A+B+D=5$, then $C=7$. It is now easily found that $A=1$ and $B=3$. Therefore

$$\frac{8x^3+19x^2+10x+5}{(x^2+2x+1)(x^2+1)} = \frac{1}{x+1} + \frac{3}{(x+1)^2} + \frac{7x+1}{x^2+1}\,.$$

6. Assess the convergence of the following integrals. If an integral converges, either evaluate it or provide an upper bound on its value.

   (a) $\displaystyle\int_1^\infty xe^{-x^2}\,dx$

   **Solution:**
   Substitute $u=x^2$ to obtain

$$\int_1^b xe^{-x^2}\,dx = \frac{1}{2}\int_1^{b^2} e^{-u}\,du = \frac{1}{2}\left[\frac{1}{e} - e^{-b^2}\right]\,.$$

And since $\lim_{b\to\infty} e^{-b^2} = 0$ we get

$$\int_1^\infty xe^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b xe^{-x^2}\,dx = \frac{1}{2e}\ .$$

Therefore the integral converges, and is equal to $\frac{1}{2e}$.

(b) $\displaystyle\int_1^\infty \frac{1}{x}\,e^{-x^2}\,dx$

**Solution**:
If $x \geq 1$, then $\frac{1}{x} \leq 1 \leq x$, from which it follows that

$$\int_1^\infty \frac{1}{x}\,e^{-x^2}\,dx \leq \int_1^\infty xe^{-x^2}\,dx\ .$$

In (a) we showed that the integral on the right was convergent, therefore the Comparison Theorem ensures that the integral on the left converges as well. And since $\int_1^\infty xe^{-x^2}\,dx = \frac{1}{2e}$, it follows that $\int_1^\infty \frac{1}{x}e^{-x^2}\,dx$ is no larger than $\frac{1}{2e}$.
Other solutions are possible here, for example we could observe that $\frac{1}{x} \leq 1$ for $x \geq 1$, and compare the given integral with $\int_1^\infty e^{-x^2}\,dx$. We could then note that $e^{-x^2} \leq e^{-x}$ for $x \geq 1$ and compare the latter integral with $\int_1^\infty e^{-x}\,dx$, which is demonstrably convergent and equal to $\frac{1}{e}$ (you would need to show this to get full credit). Thus we would be led to the same conlcusion, namely that $\int_1^\infty \frac{1}{x}e^{-x}\,dx$ converges, but would get an upper bound of $\frac{1}{e}$.

7. Evaluate $\displaystyle\int_0^3 \frac{2x}{x^2-1}\,dx$.

**Solution:**
The integrand has a vertical asymptote at $x = 1$, and therefore the integral will converge if and only if each of $\int_0^1 \frac{2x}{x^2-1}\,dx$ and $\int_1^3 \frac{2x}{x^2-1}\,dx$

converge. Checking the former first, we find

$$\begin{aligned}\int_0^1 \frac{2x}{x^2-1}\,dx &= \lim_{c\to 1^-}\int_0^c \frac{2x}{x^2-1}\,dx \\ &= \lim_{c\to 1^-}\int_0^{c^2} \frac{1}{u-1}\,du \\ &= \lim_{c\to 1^-}\left[\ln\left(1-c^2\right)\right] \\ &= -\infty\ .\end{aligned}$$

Therefore $\int_0^1 \frac{2x}{x^2-1}\,dx$ diverges, so that $\int_0^3 \frac{2x}{x^2-1}\,dx$ diverges as well.

We could have just as easily found that

$$\begin{aligned}\int_1^3 \frac{2x}{x^2-1}\,dx &= \lim_{c\to 1^+}\int_c^3 \frac{2x}{x^2-1}\,dx \\ &= \lim_{c\to 1^+}\int_{c^2}^{9} \frac{1}{u-1}\,du \\ &= \lim_{c\to 1^+}\left[\ln(8)-\ln\left(c^2-1\right)\right] \\ &= \infty\ ,\end{aligned}$$

and been led to the same conclusion.

8. Show that $\Gamma(n+1) = n\Gamma(n)$ for any integer $n \geq 1$. Be precise (i.e. carefully justify each step/calculation). You may assume that the integral defining $\Gamma(n)$ is convergent for any integer $n$.

**Solution**:
Recall that $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,dx$. Thus

$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty x^n e^{-x}\,dx \\ &= \lim_{b\to\infty}\int_0^b x^n e^{-x}\,dx\ .\end{aligned}$$

Use integration by parts with $u = x^n$ and $dv = e^{-x}\,dx$ (so that $du = nx^{n-1}dx$ and $v = -e^{-x}$) to find that

$$\begin{aligned}\int_0^b x^n e^{-x}\,dx &= -\,x^n e^{-x}\Big|_{x=0}^{x=b} + n\int_0^b x^{n-1}e^{-x}\,dx \\ &= -b^n e^{-b} + n\int_0^b x^{n-1}e^{-x}\,dx\ .\end{aligned}$$

Now in order to evaluate the limit as $b \to \infty$ observe that

- Repeated use ($n$ applications to be precise) of L'Hospital's Rule shows that $\lim_{b\to\infty} b^n e^{-b} = \lim_{b\to\infty} \frac{b^n}{e^b} = 0$.
- By definition of the improper integral, $\lim_{b\to\infty}\int_0^b x^{n-1}e^{-x}\,dx = \int_0^\infty x^{n-1}e^{-x}\,dx$.

Therefore

$$\begin{aligned}\Gamma(n+1) &= \lim_{b\to\infty}\left[-b^n e^{-b} + n\int_0^b x^{n-1}e^{-x}\,dx\right] \\ &= -\lim_{b\to\infty} b^n e^{-b} + n\lim_{b\to\infty}\int_0^b x^{n-1}e^{-x}\,dx \\ &= 0 + n\int_0^\infty x^{n-1}e^{-x}\,dx \\ &= n\Gamma(n)\ ,\end{aligned}$$

as required.

9. (a) Use the formal definition to prove that the sequence $a_n = 3 + (-1)^n\,\frac{1}{n+7}$ converges to the limit $L = 3$

**Solution**:

To begin observe that $|a_n - 3| = \frac{1}{n+7}$, which is decreasing with $n$. Also note that if $\varepsilon > 0$, then $\frac{1}{n+7} < \varepsilon$ if and only if $n > \frac{1}{\epsilon} - 7$.

Now let $\varepsilon > 0$ be given. No matter how large $\frac{1}{\epsilon} - 7$ is, there is an integer which exceeds it (for example $1 + \max\left(\lceil \frac{1}{\epsilon} - 7 \rceil, 1\right)$). Let $N$ be such an integer; that is $N$ is such that $N > \frac{1}{\epsilon} - 7$, or what is equivalent, $|a_N - 3| < \varepsilon$. If $n \geq N$, then

$$|a_n - 3| = \frac{1}{n+7} \leq \frac{1}{N+7} = |a_N - 3| < \varepsilon \ .$$

Thus for any $\varepsilon > 0$ there exists an integer $N$ for which $|a_n - 3|$ whenever $n \geq N$. Therefore $a_n$ converges to 3.

(b) Begin with the observations that (i) $a_n = \frac{n+1}{\sqrt{n}} = \sqrt{n} + \frac{1}{\sqrt{n}} > \sqrt{n}$ and (ii) for $M > 0$, $\sqrt{n} > M$ if and only if $n > M^2$.

Now let $M > 0$ be given. No matter how large $M^2$, there is an integer which exceeds it ($\lceil M^2 \rceil$ for example). Let $N$ be such an integer; that is $N$ is such that $N > M^2$, equivalently $\sqrt{N} > M$. If $n \geq N$, then

$$a_n > \sqrt{n} \geq \sqrt{N} > M \ .$$

Thus for all $M > 0$ there exists an integer $N$ for which $a_n > M$ whenever $n \geq N$. Therefore $\lim_{n\to\infty} a_n = \infty$.

10. Determine whether or not the following sequences converge (you do not need to use the formal definition). If a sequence converges, evaluate its limit (state any theorems you use along the way). If a sequence diverges, explain why.

(a) $a_n = n^{1/n}$.

**Solution**:

This is of the form $\infty^0$, which is indeterminate. So let $b_n = \ln(a_n) = \frac{\ln(n)}{n}$. Using L'Hopital's Rule we get that $b_n \to 0$, and since $a_n = e^{b_n}$ and $f(x) = e^x$ is continuous at $x = 0$, we get that $a_n$ converges to $e^0 = 1$.

(b) $a_n = \dfrac{1}{\sin\left(\frac{(-1)^n}{n}\right)}$.

**Solution:**
The sequence $\frac{(-1)^n}{n}$ converges to zero, and $\sin x$ is continuous at $x = 0$, therefore $\sin\left(\frac{(-1)^n}{n}\right)$ converges to zero as well. However if $n$ is even, then $\sin\left(\frac{(-1)^n}{n}\right) = \sin\left(\frac{1}{n}\right) > 0$, and if $n$ is odd then $\sin\left(\frac{(-1)^n}{n}\right) = \sin\left(-\frac{1}{n}\right) < 0$. Thus the even terms of our sequence will diverge to $\infty$, whereas the odd terms will diverge to $-\infty$. Therefore the sequence is divergent; note in particular that it is *not* true that $\lim_{n\to\infty} a_n = \infty$.

(c) $a_n = \dfrac{1 + \cos(n)}{\ln(n)}$.

**Solution:**
Since $-1 \leq \cos(n) \leq 1$, we have $0 \leq 1 + \cos(n) \leq 2$. Therefore $0 \leq a_n \leq \frac{2}{\ln(n)}$, and since $\frac{2}{\ln(n)}$ converges to zero, so does $a_n$ by the Squeeze Theorem.