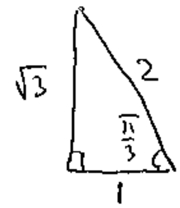
1.

5 marks (a) Evaluate $\displaystyle\int_0^3 \frac{\sqrt{x}}{x^2+x}\,dx$.

Improper integral of type 2



$$\int_0^3 \frac{\sqrt{x}}{x^2+x}\,dx = \lim_{a\to 0+} \int_a^3 \frac{\sqrt{x}}{x^2+x}\,dx$$

Substitution $u=\sqrt{x}$

$du = \frac{1}{2}x^{-1/2}dx$

$dx = 2\sqrt{x}\,du$

$$\int_a^3 \frac{\sqrt{x}}{x^2+x}dx = \int_{\sqrt{a}}^{\sqrt{3}} \frac{u}{u^4+u^2}\,2\sqrt{x}\,du = 2\int_{\sqrt{a}}^{\sqrt{3}} \frac{u^2}{u^4+u^2}du$$

$$= 2\int_{\sqrt{a}}^{\sqrt{3}} \frac{1}{u^2+1}du = 2\left[\arctan u\right]_{\sqrt{a}}^{\sqrt{3}} = 2(\arctan\sqrt{3} - \arctan\sqrt{a})$$

$$= 2\left(\frac{\pi}{3} - \arctan\sqrt{a}\right)$$

$$= \lim_{a\to 0+} 2\left(\frac{\pi}{3} - \arctan\sqrt{a}\right)$$

$$= 2\left(\frac{\pi}{3} - \arctan 0\right) = \boxed{\frac{2\pi}{3}}$$

5 marks (b) Integrate $\displaystyle\int x\tan^{-1}x\,dx$.

Integration by parts

$$\int \underbrace{\tan^{-1}x}_{f}\ \underbrace{x}_{g'}\,dx = \tan^{-1}x\ \frac{x^2}{2} - \int (\tan^{-1}x)'\,\frac{x^2}{2}dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{1+x^2}dx$$

$g(x) = \frac{x^2}{2}$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \boxed{\frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x + C}$$

10 marks 2. Evaluate $\int_0^\infty e^{-ax}\cos bx\,dx$, where a and b are positive constants. Carefully explain your work.

To find $\int e^{-ax}\cos(bx)\,dx$ we integrate by parts twice.

$$\int \underbrace{e^{-ax}}_{f}\underbrace{\cos(bx)}_{g'}\,dx = e^{-ax}\frac{\sin(bx)}{b} - \int(-ae^{-ax})\frac{\sin(bx)}{b}\,dx$$

$g(x) = \frac{\sin(bx)}{b}$

$$= \frac{1}{b}e^{-ax}\sin(bx) + \frac{a}{b}\int \underbrace{e^{-ax}}_{f}\underbrace{\sin(bx)}_{g'}\,dx$$

$g(x) = -\frac{\cos(bx)}{b}$

$$= \frac{1}{b}e^{-ax}\sin(bx) + \frac{a}{b}\left[e^{-ax}\left(-\frac{\cos(bx)}{b}\right) - \int(-ae^{-ax})\left(-\frac{\cos(bx)}{b}\right)dx\right]$$

$$= \frac{1}{b}e^{-ax}\sin(bx) - \frac{a}{b^2}e^{-ax}\cos(bx) - \frac{a^2}{b^2}\int e^{-ax}\cos(bx)\,dx$$

$$\Rightarrow \int e^{-ax}\cos(bx)\,dx = \frac{1}{1+\frac{a^2}{b^2}}\,e^{-ax}\left(\frac{1}{b}\sin(bx) - \frac{a}{b^2}\cos(bx)\right) + C$$

$$\int_0^\infty e^{-ax}\cos(bx)\,dx = \lim_{t\to\infty}\int_0^t e^{-ax}\cos(bx)\,dx = \lim_{t\to\infty}\left[\frac{1}{1+\frac{a^2}{b^2}}e^{-ax}\left(\frac{1}{b}\sin(bx) - \frac{a}{b^2}\cos(bx)\right)\right]_0^t$$

improper integral of type 1

$$= \frac{1}{1+\frac{a^2}{b^2}}\lim_{t\to\infty}\left[e^{-at}\left(\frac{1}{b}\sin(bt) - \frac{a}{b^2}\cos(bt)\right) - \left(\frac{1}{b}\sin 0 - \frac{a}{b^2}\cos 0\right)\right]$$

$$= \frac{1}{1+\frac{a^2}{b^2}}\left(0 + \frac{a}{b^2}\right) = \boxed{\frac{a}{b^2+a^2}}$$

because $a > 0$

10 marks 3. Evaluate $\int_0^1 \ln x\, dx$. Carefully explain your work.

Improper integral of type 2

$$\int_0^1 \ln x\, dx = \lim_{a\to 0+} \int_a^1 \ln x\, dx$$

$$\int_a^1 \ln x\, dx = \int_a^1 \underbrace{\ln x}_{f} \cdot \underbrace{1}_{g'}\, dx = \left[(\ln x)x\right]_a^1 - \int_a^1 (\ln x)' x\, dx$$

integration by parts

$g(x) = x$

$$= \ln 1 - a\ln a - \int_a^1 \frac{1}{x} x\, dx$$

$$= -a\ln a - [x]_a^1 = -a\ln a - (1-a) = -a\ln a - 1 + a$$

$$= \lim_{a\to 0+} (-a\ln a - 1 + a) = 0 - 1 + 0 = \boxed{-1}$$

because

$$\lim_{x\to 0+} x\ln x = \lim_{x\to 0+} \frac{\ln x}{1/x} = \lim_{x\to 0+} \frac{(\ln x)'}{(1/x)'} = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$0\cdot\infty$ $\qquad \frac{\infty}{\infty}$

$$= \lim_{x\to 0+} (-x) = 0$$

4.

4 marks (a) Give the definition of the Gamma function $\Gamma(x)$.

Note Gamma function is not on the 2014 Midterm #1 (Feb.7)

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt \quad , x>0$$

6 marks (b) Calculate directly, without using any additional properties, the value of $\Gamma(2)$.

$$\Gamma(2) = \int_0^\infty t^{2-1} e^{-t}\,dt = \int_0^\infty t e^{-t}\,dt = \lim_{b\to\infty} \int_0^b t e^{-t}\,dt$$

integration by parts

$$\int_0^b \underbrace{t}_{f} \underbrace{e^{-t}}_{g'}\,dt = \left[t(-e^{-t})\right]_0^b - \int_0^b (-e^{-t})\,dt = -be^{-b} + \left[-e^{-t}\right]_0^b$$

$g(t) = -e^{-t}$

$$= -be^{-b} - e^{-b} + 1$$

$$= \lim_{b\to\infty} \left(-be^{-b} - e^{-b} + 1\right) = 0 + 0 + 1 = \boxed{1}$$

because

$$\lim_{x\to\infty} (xe^{-x}) = \lim_{x\to\infty} \underbrace{\frac{x}{e^x}}_{\frac{\infty}{\infty}} = \lim_{x\to\infty} \frac{x'}{(e^x)'} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

5.

6 marks (a) Give the $\epsilon - \delta$ definition of continuity of a function $f(x)$ at a point x_0.

$f(x)$ is <u>continuous at x_0</u> if it is defined near x_0

and $\lim_{x \to x_0} f(x) = f(x_0)$

i.e. $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

4 marks (b) Use part (a) to prove that the function $f(x) = 3(x-1)^2$ is continuous at the point $x_0 = 1$.

$f(1) = 0$ We need to show: $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$|x-1| < \delta \Rightarrow |3(x-1)^2 - 0| < \varepsilon$$

$$3|x-1|^2 < \varepsilon$$

$$|x-1|^2 < \frac{\varepsilon}{3}$$

$$|x-1| < \sqrt{\frac{\varepsilon}{3}}$$

<u>Proof.</u> take arbitrary $\varepsilon > 0$.

Let $\delta = \sqrt{\frac{\varepsilon}{3}}$

Then for any x s.t. $|x-1| < \delta$ we have:

$$|f(x) - f(1)| = |3(x-1)^2 - 0| = 3|x-1|^2 < 3\delta^2 = 3\frac{\varepsilon}{3} = \varepsilon$$

$\Rightarrow f$ is continuous at 1 ∎

6.

5 marks (a) State the Mean Value Theorem

Suppose $a<b$,
a function $f(x)$ is continuous on $[a,b]$,
differentiable on (a,b).

Then $\exists\, c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

5 marks (b) Use the Mean Value Theorem to prove the inequality $xe \leq e^x$ for all $x > 1$.

Apply MVT to $f(x) = e^x - ex$ on $[1,b]$ for $b>1$.

f is continuous on $[1,b]$, differentiable on $(1,b)$
(since it's differentiable on $(-\infty,\infty)$).

$f'(x) = e^x - e > 0$ for $x>1$.

By MVT $\exists\, c \in (1,b)$ s.t. $f'(c) = \dfrac{f(b)-f(1)}{b-1}$

$f'(c) > 0$

$\dfrac{f(b)-f(1)}{b-1} > 0$

$b-1>0 \Rightarrow f(b)-f(1) > 0$

$e^b - eb - (e-e) > 0$

$e^b - eb > 0 \Rightarrow be \leq e^b$ for $b>1$.

10 marks 7. Use part (b) of Problem 6 to determine whether the improper integral $\int_1^{\infty} \frac{dx}{x^2 - xe + e^x}\,dx$ converges or not. Do not evaluate the integral.

$xe \le e^x$ for $x \ge 1$ (for $x>1$ #6(b), for $x=1$ $e=e$)

$-xe+e^x \ge 0$ for $x \ge 1$

$\Rightarrow \quad 0 \le \frac{1}{x^2-xe+e^x} \le \frac{1}{x^2}$ for $x \ge 1$

continuous on $[1,\infty)$

$\int_1^{\infty} \frac{1}{x^2}\,dx$ converges $\underset{\text{Comparison Theorem}}{\Longrightarrow}$ $\int_1^{\infty} \frac{dx}{x^2-xe+e^x}\,dx$ converges

$//$

$\lim_{b\to\infty} \int_1^b x^{-2}\,dx = \lim_{b\to\infty} \left[\frac{x^{-2+1}}{-2+1}\right]_1^b$

$= \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = -\lim_{b\to\infty}\left(\frac{1}{b}-1\right) = -(0-1) = 1$

10 marks 8. Write out the form of the partial fraction decomposition of the function

$$\frac{2x^2+6x-7}{(x^2+x-2)^2\,(x^2+6x+13)^2}.$$

Do not evaluate the coefficients.

$x^2+6x+13$ is irreducible $\quad 6^2-4\times 13 = 36-52<0$

$x^2+x-2 = (x+2)(x-1)$

$$\frac{2x^2+6x-7}{(x+2)^2(x-1)^2(x^2+6x+13)^2} =$$

$$\frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{Ex+F}{x^2+6x+13} + \frac{Gx+H}{(x^2+6x+13)^2}$$