10 marks 1. If $a_n = (-1)^n \frac{n}{n+1}$, is $\{a_n\}_{n=1}^{\infty}$ monotonic? Is it bounded? Explain.

Since $a_n > 0$ for n even and $a_n < 0$ for n odd the sequence is not monotonic.

Observe that $|a_n| = \frac{n}{n+1} < 1$ so $-1 < a_n < 1$ and $\{a_n\}$ is bounded.

5 marks 2. Let $a_n = \dfrac{1}{2^n}$. Use the formal ε, N definition to show that $\lim_{n\to\infty} a_n = 0$.

Given $\varepsilon > 0$, let $N \geq \max\left\{ -\frac{\ln \varepsilon}{\ln 2}, 1 \right\}$.

Then $n > N \Rightarrow n > -\dfrac{\ln \varepsilon}{\ln 2}$

$$\Rightarrow n \ln 2 > \ln \frac{1}{\varepsilon}$$

$$\Rightarrow 2^n > \frac{1}{\varepsilon}$$

$$\Rightarrow \frac{1}{2^n} < \varepsilon$$

$$\Rightarrow \left| \frac{1}{2^n} - 0 \right| < \varepsilon .$$

QED

5 marks 3. Use the Squeeze Theorem to determine $\lim_{n\to\infty} \frac{n+\ln n}{n^2+(-1)^n}$.

Since $n > \ln n$ for $n > 0$

$$0 < \frac{n+\ln n}{n^2+(-1)^n} < \frac{2n}{n^2-1} \quad \text{for } n > 1$$

$$\lim_{n\to\infty} \frac{2n}{n^2-1} = \lim_{x\to\infty} \frac{2x}{x^2-1} = \lim_{x\to\infty} \frac{\frac{2x}{x^2}}{\frac{x^2-1}{x^2}} = \lim_{x\to\infty} \frac{\frac{2}{x}}{1-\frac{1}{x^2}} = \frac{0}{1-0} = 0$$

$\Rightarrow$ by the Squeeze Theorem

$$\lim_{n\to\infty} \frac{n+\ln n}{n^2+(-1)^n} = 0$$

4.

5 marks (a) Give example of a series $\sum_{n=1}^{\infty} a_n$ such that $\lim_{n\to\infty} a_n = 0$, but the series diverges.

The harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$

5 marks (b) Let $\sum_{n=1}^{\infty} a_n$ be a series with only positive terms, and let $S_N = \sum_{n=1}^{N} a_n$ be the partial sum of the first N terms of the series (i.e., the partial sum of order N). Prove that if $S_N < 5 - \sin(N^2)$, then the series $\sum a_n$ converges.

Since $a_n > 0$ the sequence $\{S_N\}_{N=1}^{\infty}$ is increasing. Since $S_N < 5 - \sin(N^2) \leq 6$ $\{S_N\}_{N=1}^{\infty}$ is bounded above. By the Monotone Convergence Theorem $\{S_N\}$ converges and, by definition, this implies $\sum a_n$ converges.

5. Determine if each of the following series is convergent or divergent. Be clear about any test for convergence/divergence you apply.

5 marks (a) $\sum_{n=1}^{\infty} \frac{3^{n^2}}{n!}$

Ratio Test

$$\lim_{n\to\infty}\left|\frac{3^{(n+1)^2}}{(n+1)!}\frac{n!}{3^{n^2}}\right| = \lim_{n\to\infty}\frac{3^{2n+1}}{n+1} = \lim_{x\to\infty}\frac{3^{2x+1}}{x+1} \underset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty}\frac{(3^{2x+1})'}{(x+1)'}$$

$$= \lim_{x\to\infty}\frac{3^{2x+1}(\ln 3)2}{1} = \infty$$

The series diverges

5 marks (b) $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ The series is convergent

This is example 3 (11.4 p. 729)

6. Determine if each of the following series converges absolutely, converges conditionally, or diverges. Be clear about any test for convergence/divergence you apply.

5 marks (a) $\sum_{n=1}^{\infty}(-1)^n \frac{n}{1+\ln n}$

$\lim_{n\to\infty}(-1)^n \frac{n}{1+\ln n}$ is not zero

because $\lim_{n\to\infty} \frac{n}{1+\ln n} = \lim_{x\to\infty} \frac{x}{1+\ln x} \underset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{x'}{(1+\ln x)'} = \lim_{x\to\infty} \frac{1}{0+\frac{1}{x}}$

$= \lim_{x\to\infty} x = \infty$

$\Rightarrow$ the series diverges by the Test for Divergence

5 marks (b) $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{(\ln n)^2}{n}$ $\quad b_n = \frac{(\ln n)^2}{n} > 0$ for all n

alternating series

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{(\ln n)^2}{n} = \lim_{x\to\infty} \frac{(\ln x)^2}{x} \underset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{((\ln x)^2)'}{x'} = \lim_{x\to\infty} \frac{2(\ln x)\frac{1}{x}}{1} \underset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} 2\frac{(\ln x)'}{x'}$

$= \lim_{x\to\infty} 2\frac{\frac{1}{x}}{1} = 0$

$b_n = f(n)$ for $f(x) = \frac{(\ln x)^2}{x}$ $\quad f'(x) = \frac{2(\ln x)\frac{1}{x}x - (\ln x)^2}{x^2} = \frac{\ln x(2-\ln x)}{x^2} < 0$

for $2 - \ln x < 0$ $\quad \ln x > 2$

$x > e^2$

$\Rightarrow b_{n+1} \le b_n$ for sufficiently large n

$\underset{AST}{\Rightarrow}$ the series converges

Now we ask if the series is absolutely convergent. One way to tell is to use the comparison theorem.

If $n > e$ then $\ln n > 1$

$$\Rightarrow (\ln n)^2 > 1$$

$$\Rightarrow \frac{(\ln n)^2}{n} > \frac{1}{n} > 0$$

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series), and so by the comparison theorem $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n}$ diverges.

It follows that the given series converges conditionally.

10 marks 7. Use the power series representation of $\frac{1}{1-x}$ to evaluate $\sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^{n-1}$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad |x|<1$$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \sum_{n=1}^{\infty} n\,x^{n-1} \quad \text{for} \quad |x|<1$$

$\frac{1}{3}$ lies within the radius of convergence.

So,

$$\sum_{n=1}^{\infty} n\left(\frac{1}{3}\right)^{n-1} = \frac{d}{dx}\left(\frac{1}{1-x}\right)\Bigg|_{x=\frac{1}{3}}$$

$$= \frac{1}{(1-\frac{1}{3})^2} = \frac{9}{4}$$

10 marks 8. Given $\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n}}(x-1)^n$, what is the interval of convergence of the series?

Ratio Test :

$$\left| \frac{(x-1)^{n+1}}{2^{n+1}\sqrt{n+1}} \cdot \frac{2^n\sqrt{n}}{(x-1)^n} \right| = \frac{|x-1|}{2} \frac{\sqrt{n}}{\sqrt{n+1}} \xrightarrow{n\to\infty} \frac{|x-1|}{2}$$

$$\frac{|x-1|}{2} < 1 \implies |x-1| < 2 \quad \text{so } R=2$$

Check Endpts :

$x=3$ $\quad \sum_{n=1}^{\infty} \frac{2^n}{2^n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges (it's a p-series with $p \le 1$)

$x=-1$ $\quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

$b_n = \frac{1}{\sqrt{n}}$, $b_n \to 0$ as $n \to \infty$

$n+1 > n$

$\therefore \sqrt{n+1} > \sqrt{n}$

$\therefore \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$

$\therefore b_{n+1} < b_n$

$\therefore$ converges by Alternating Series Test.

Interval of Convergence is $[-1, 3)$

9.

5 marks (a) Evaluate $\int e^{-x^2}\,dx$ as a power series centred at the origin. Write the first three nonzero terms of the series.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$

for all x (see part b).

$$\int e^{-x^2}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n}\,dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{x^{2n+1}}{(2n+1)} = C + x - \frac{x^3}{3} \ldots$$

5 marks (b) Determine the interval of convergence of the power series found in Part (a) above.

Ratio test applied to $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}\, x^{2n+2}}{(n+1)!} \cdot \frac{n!}{(-1)^n x^{2n}} \right|$$

$$= \lim_{n\to\infty} \frac{x^2}{n+1} = 0 \quad \forall \; x \in \mathbb{R}$$

It follows that the radius of convergence is $R = \infty$ and $(-\infty, \infty)$ is the interval of convergence.