5 *marks* 1.

(a) $\Gamma(3)+\Gamma(4) = 2!+3! = 2+2\cdot 3 = \boxed{8}$

(b) If $x_0 > 1$ is such that $\Gamma(x_0) = \frac{3}{2}$, then

$$\frac{\Gamma(x_0+1)}{2x_0} = \frac{x_0\,\Gamma(x_0)}{2x_0} = \frac{\Gamma(x_0)}{2}$$

$$= \frac{1}{2}\,\frac{3}{2} = \boxed{\frac{3}{4}}$$

4 *marks* 2. Prove that if $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then

$$\lim_{n\to\infty}(a_n b_n) = 0.$$

This is a problem from the list of suggested exercises
#89 (11.1) p. 702

Proof: Since $\{b_n\}$ is bounded, there is a positive number M such that $|b_n| \le M$ and hence, $|a_n|\,|b_n| \le |a_n|\,M$ for all $n \ge 1$. Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = 0$, there is an integer N such that $|a_n - 0| < \frac{\varepsilon}{M}$ if $n > N$. Then $|a_n b_n - 0| = |a_n b_n| = |a_n|\,|b_n| \le |a_n|\,M = |a_n - 0|\,M < \frac{\varepsilon}{M}\cdot M = \varepsilon$ for all $n > N$. Since ε was arbitrary, $\lim_{n\to\infty}(a_n b_n) = 0$.

4 marks 3. Use the Integral Test to determine whether the series is convergent or divergent:

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$$

Let $f(x) = \frac{\ln x}{x^2} > 0$, continuous on $[2, \infty)$

decreasing on $[2,\infty)$ because $f'(x) = \frac{\frac{1}{x}x^2 - (\ln x)2x}{x^4} = \frac{1-2\ln x}{x^3} < 0$

for $\ln x > \frac{1}{2}$

$x > \sqrt{e}$

$f(n) = \frac{\ln n}{n^2}$

$$\int_2^\infty \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\int_2^b \frac{\ln x}{x^2}dx = \lim_{b\to\infty}\int_2^b \underbrace{\ln x}_{u}\,\underbrace{\frac{1}{x^2}}_{v'}\,dx = \lim_{b\to\infty}\left[\left[\ln x\left(-\frac{1}{x}\right)\right]_2^b - \int_2^b \frac{1}{x}\left(-\frac{1}{x}\right)dx\right]$$

$v = -\frac{1}{x}$

$$= \lim_{b\to\infty}\left[-\frac{\ln b}{b} + \frac{\ln 2}{2} + \left[\frac{x^{-2+1}}{-2+1}\right]_2^b\right] = \lim_{b\to\infty}\left(0 + \frac{\ln 2}{2} - \frac{1}{b} + \frac{1}{2}\right) = \frac{\ln 2}{2} + \frac{1}{2}$$

$\frac{\infty}{\infty}$ $\frac{(\ln b)'}{b'} = \frac{1/b}{1} = \frac{1}{b} \to 0$ by l'Hopital's rule

The integral converges $\underset{IT}{\Longrightarrow}$ the series is convergent.

7 marks

4. Determine whether the series converges or diverges. If it converges, - find its sum.

$$\text{(a)} \sum_{n=3}^{\infty} (-\frac{1}{8})^n$$

This is a geometric series with ratio $r = -\frac{1}{8}$

$|r| = \frac{1}{8} < 1 \Rightarrow$ the series converges

$$\sum_{n=3}^{\infty}\left(-\frac{1}{8}\right)^n = \left(-\frac{1}{8}\right)^3 + \left(-\frac{1}{8}\right)^4 + \left(-\frac{1}{8}\right)^5 + \dots = \left(-\frac{1}{8}\right)^3\left(1 + \left(-\frac{1}{8}\right) + \left(-\frac{1}{8}\right)^2 + \left(-\frac{1}{8}\right)^3 + \dots\right)$$

$$= \left(-\frac{1}{8}\right)^3 \frac{1}{1-\left(-\frac{1}{8}\right)} = \left(-\frac{1}{8}\right)^3 \frac{1}{1+\frac{1}{8}} = -\frac{1}{8^3}\,\frac{1}{\frac{9}{8}} = \boxed{-\frac{1}{8^2 \times 9}}$$

$$\text{(b)} \sum_{n=1}^{\infty} \frac{n\arctan(\sqrt{n}+3)}{n+1}$$

$$\lim_{n\to\infty} \frac{n\arctan(\sqrt{n}+3)}{n+1} = \lim_{n\to\infty} \frac{\arctan(\sqrt{n}+3)}{1+\frac{1}{n}} = \frac{\frac{\pi}{2}}{1+0} = \frac{\pi}{2} \neq 0$$

$\Rightarrow$ the series diverges

8 marks 5. Determine whether the series is convergent or divergent. State clearly which test you are using.

$$\text{(a)} \sum_{n=1}^{\infty} \frac{1}{(6n^5+23)^{1/4}}$$

$0 < \frac{1}{(6n^5+23)^{1/4}} < \frac{1}{n^{5/4}}$ for all n

$\sum_{n=1}^{\infty} \frac{1}{n^{5/4}}$ converges (p-series with $p = \frac{5}{4} > 1$)

$\Rightarrow$ by the Comparison Test

$\sum_{n=1}^{\infty} \frac{1}{(6n^5+23)^{1/4}}$ is convergent too

Note:

$6n^5+23 > n^5$

$(6n^5+23)^{1/4} > n^{5/4}$

$\frac{1}{n^{5/4}} > \frac{1}{(6n^5+23)^{1/4}}$

$$\text{(b)} \sum_{n=1}^{\infty} \frac{n^2-4n+5}{10n^3+n^2+4n}$$

Apply Limit Comparison Test. Set $a_n = \frac{n^2-4n+5}{10n^3+n^2+4n}$

$b_n = \frac{1}{n}$

$a_n > 0, b_n > 0$ for all n

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{10} > 0$

$\sum_{n=1}^{\infty} b_n$ diverges (the harmonic series)

$\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges too.

8 marks 6. Use the Alternating Series Test (if possible) to test the following series for convergence:

$$\text{(a)} \sum_{n=1}^{\infty} (-1)^n \frac{2n^2+5n}{4n^2+6n+1}$$

$\lim_{n\to\infty} \frac{2n^2+5n}{4n^2+6n+1} = \frac{1}{2} \neq 0 \Rightarrow$ AST is not applicable

$\lim_{n\to\infty} (-1)^n \frac{2n^2+5n}{4n^2+6n+1}$ DNE $(\Rightarrow \neq 0) \Rightarrow$ the series diverges

$$\text{(b)} \sum_{n=1}^{\infty} (-1)^n n e^{-n}$$

Set $b_n = ne^{-n}$

$b_n > 0$ for all n

Let $f(x) = xe^{-x}$ $\quad f(n) = b_n$

$\lim_{n\to\infty} b_n = 0$ because $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{e^x} \underset{\text{l'Hopital's rule}}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$

$f(x)$ is decreasing because $f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) < 0$ for $x > 1$.
for $x \geqslant 1$

$\Rightarrow \{b_n\}$ is decreasing.

$\underset{AST}{\Rightarrow}$ the series converges.

12 marks 7. Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent.

$$\text{(a)} \sum_{n=1}^{\infty} (-1)^n \frac{-1}{\sqrt{n+5}}$$

This series is conditionally convergent.

Indeed, it is convergent by AST (because $\left\{\frac{1}{\sqrt{n+5}}\right\}$ decreases to zero)

and it is not absolutely convergent since

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+5}}$ diverges (proof: apply LCT $a_n = \frac{1}{\sqrt{n+5}} > 0$

$b_n = \frac{1}{\sqrt{n}} > 0$

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$

$\sum_{n\geq 1}^{\infty} b_n$ diverges as p-series with $p = \frac{1}{2} \leq 1$

$\Rightarrow \sum_{n\geq 1}^{\infty} a_n$ diverges)

$$\text{(b)} \sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{n+1}\right)^{n^2}$$

Apply the Root Test

$$\lim_{n\to\infty} \left|(-1)^n\left(\frac{n}{n+1}\right)^{n^2}\right|^{1/n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$$

$\Rightarrow$ the series is absolutely convergent

$$\text{(c)} \sum_{n=1}^{\infty} \frac{n^{55}\, 25^n}{n!}$$

Use the Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^{55} 25^{n+1}}{(n+1)!}}{\frac{n^{55} 25^n}{n!}} \right| = \lim_{n\to\infty} \frac{(n+1)^{55}}{n^{55}} \frac{25}{n+1} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{55} \frac{25}{n+1}$$

$$\text{or } \lim_{n\to\infty} \frac{(n+1)^{54}}{n^{55}} 25 = 0 < 1$$

$\Rightarrow$ the series is absolutely convergent

12 marks 8.

Find the radius and the interval of convergence for the following series;

$$\text{(a)} \sum_{n=1}^{\infty} \frac{4^n}{\sqrt{n}}(x-5)^n$$

Apply Ratio Test:

$$\lim_{n\to\infty}\left|\frac{\frac{4^{n+1}}{\sqrt{n+1}}(x-5)^{n+1}}{\frac{4^n}{\sqrt{n}}(x-5)^n}\right| = 4|x-5|<1 \qquad |x-5|<\frac{1}{4} \qquad \Rightarrow \boxed{R=\frac{1}{4}}$$

Endpoints: $x=5-\frac{1}{4}$ $\quad \sum_{n=1}^{\infty}\frac{4^n}{\sqrt{n}}\left(-\frac{1}{4}\right)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by AST

$x=5+\frac{1}{4}$ $\quad \sum_{n=1}^{\infty}\frac{4^n}{\sqrt{n}}\left(\frac{1}{4}\right)^n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges (p-series, $p=\frac{1}{2}\le 1$)

$$\boxed{I=\left[5-\frac{1}{4},\ 5+\frac{1}{4}\right)}$$

$$\text{(b)} \sum_{n=1}^{\infty} \frac{(3x-1)^n}{6^n\sqrt{n}}$$

Apply Ratio Test:

$$\lim_{n\to\infty}\left|\frac{\frac{(3x-1)^{n+1}}{6^{n+1}\sqrt{n+1}}}{\frac{(3x-1)^n}{6^n\sqrt{n}}}\right| = \frac{|3x-1|}{6}<1 \qquad |3x-1|<6 \qquad |x-\tfrac{1}{3}|<2 \qquad \boxed{R=2}$$

Endpoints: $x=\frac{1}{3}-2$ $\quad \sum_{n=1}^{\infty}\frac{(-6)^n}{6^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges by AST

$x=\frac{1}{3}+2$ $\quad \sum_{n=1}^{\infty}\frac{6^n}{6^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges (p-series, $p=\frac{1}{2}\le 1$)

$$\boxed{I=\left[\frac{1}{3}-2,\ \frac{1}{3}+2\right)}$$

$$\text{(c)} \sum_{n=1}^{\infty} \frac{n^2(x+1)^n}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}$$

Apply Ratio Test:

$$\lim_{n\to\infty}\left|\frac{\frac{(n+1)^2(x+1)^{n+1}}{2\cdot 4\cdot 6\cdot \ldots \cdot (2(n+1))}}{\frac{n^2(x+1)^n}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{n^2}\frac{|x+1|}{2n+2} = \underset{1}{\overset{0}{\wedge}} \quad \text{for any } x$$

$$\Rightarrow \boxed{\begin{array}{l} R=\infty \\ I=(-\infty,\infty) \end{array}}$$