THE UNIVERSITY OF WESTERN ONTARIO
LONDON CANADA
DEPARTMENT OF MATHEMATICS

Calculus 1501B Final Examination

Saturday, April 25, 2015 7:00 – 10:00 p.m.

1. Let $T_3(x)$ denote the degree 3 Taylor polynomial of $f(x) = \sin x$ centered at $a = \frac{\pi}{6}$. Choose the one answer which most precisely describes the value of $T_3(\frac{\pi}{4})$.

   (a) $T_3(\frac{\pi}{4})$ approximates $\sin(\frac{\pi}{4})$ with error less than 10^{-1}.
   (b) $T_3(\frac{\pi}{4})$ approximates $\sin(\frac{\pi}{4})$ with error less than 10^{-2}.
   (c) $T_3(\frac{\pi}{4})$ approximates $\sin(\frac{\pi}{4})$ with error less than 10^{-3}.
   (d) $T_3(\frac{\pi}{4}) = \sin(\frac{\pi}{4})$.

2. Find the length of the parametric curve

$$x = 3\cos t - \sin t, \quad y = 3\sin t + \cos t, \quad t \in [0, \pi]\,.$$

   (a) $\frac{\pi}{2}$.
   (b) $\sqrt{3} + 2\pi$.
   (c) $\pi\sqrt{10}$.
   (d) $\frac{10\pi}{3}$.
   (e) None of the above.

3. Find the sum of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)2^n}$.

   (a) $-\frac{\ln(1/2)}{2}$.
   (b) 0.
   (c) $2\ln(3/2)$.
   (d) $e^{-1/2}$.
   (e) The series diverges.

4. Consider the function

$$f(x) = \begin{cases} \frac{x+1}{x+3} & \text{if } x \le 1 \\ 1 - \frac{x^2}{2} & \text{if } x > 1\,. \end{cases}$$

   Select ALL true statements below.

   (a) $f(x)$ satisfies the assumptions of the Mean Value Theorem on the interval $[-5, 0]$.
   (b) $f(x)$ satisfies the assumptions of the Mean Value Theorem on the interval $[1, 10]$.
   (c) $f(x)$ satisfies the assumptions of the Mean Value Theorem on the interval $[-10, -5]$.
   (d) $f(x)$ satisfies the assumptions of Rolle's Theorem on the interval $[1, 5]$.
   (e) $f(x)$ satisfies the assumptions of Rolle's Theorem on the interval $[-2, 2]$.

5. Let C be the polar curve $\{r = 4\cos(3\vartheta);\ 0 \le \vartheta \le 2\pi\}$. Select ALL true statements below.

   (a) C has no points of intersection with the polar curve $\{r = 3\cos\vartheta;\ 0 \le \vartheta \le \pi\}$.

   (b) C has precisely two points of intersection with $\{r = 3\cos\vartheta;\ 0 \le \vartheta \le \pi\}$.

   (c) C has precisely three points of intersection with $\{r = 3\cos\vartheta;\ 0 \le \vartheta \le \pi\}$.

   (d) C has precisely one tangent line at the origin.

   (e) C has precisely two different tangent lines at the origin.

   (f) C has precisely three different tangent lines at the origin.

   (g) C has precisely two loops (or petals).

   (h) C has precisely three loops (or petals).

   (i) C has precisely six loops (or petals).

6. Let $y = f(t)$ be the unique solution of the initial value problem

$$t\frac{\mathrm{d}y}{\mathrm{d}t} + y = \cos t\,, \qquad y(\pi) = 0\,.$$

   Select ALL true statements below.

   (a) $\lim_{t\to 0+} f(t) = 0$.

   (b) $\lim_{t\to 0+} f(t) = 1$.

   (c) $\lim_{t\to 0+} f(t)$ does not exist.

   (d) $f'(\pi) = -\pi^{-1}$.

   (e) $f'(\pi) = 0$.

   (f) $f'(\pi) = \pi^2/4$.

   (g) $\lim_{t\to\infty} f(t) = -\infty$.

   (h) $\lim_{t\to\infty} f(t) = 0$.

   (i) $\lim_{t\to\infty} f(t) = \infty$.

7. Evaluate $\displaystyle\int_1^2 \frac{2}{\sqrt[3]{2x-3}}\,\mathrm{d}x$, if possible.

8. Prove that a convergent sequence is bounded. [Hint: Use the $\varepsilon - N$ definition of convergence of a sequence.]

9. Find the Taylor series of $f(x) = (x-1)e^{(x-1)^2}$ centered at $a = 1$.

10. (a) Find the Maclaurin series of $f(x) = \cos(\pi x)$.

    (b) Determine the radius of convergence of this series.

    (c) Use Taylor's Inequality to prove that $f(x)$ is the sum of its Maclaurin series.

11. Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x(y^2+1)\cos x}{y}\,, \qquad y(0) = 1\,.$$

   Express your answer in the form $y^2 = F(x)$.

12. Let C be a spiral defined by the polar equation $r = e^{2\vartheta}$, where $\vartheta \in [0, \pi]$.

    (a) Find the length of C.

    (b) Find the equation of the tangent line to C at its terminal point. (The equation should be in the form $y = mx + b$.)

13. (a) Prove that the curve defined by the polar equation

$$r = \tan\vartheta \sec\vartheta$$

    is symmetric about the vertical line $\vartheta = \pi/2$.

    (b) Find a Cartesian equation for this curve.

14. Let C be the parametric curve defined by

$$x = e^{\sin t}, \quad y = e^{\cos t}, \quad 0 \le t \le 2\pi\,.$$

    (a) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

    (b) Determine the points where the tangent line to C is horizontal or vertical.