Question 1.

(a) State the ε–δ definition for the limit of a function at a point.

(b) Use this defition to prove that
$$\lim_{x\to 3} \frac{4x-2}{5} = 2.$$

Solution.

(a) The limit of a function f as x approaches a (here, we assume that the domain of f contains some interval around a, but not necessarily a itself) is a number L such that for every $\varepsilon > 0$, there is $\delta > 0$ such that for every x (in the domain of f)
$$\text{if } 0 < |x-a| < \delta, \text{ then } |f(x) - L| < \varepsilon.$$

(b) Observe that
$$\left|\frac{4x-2}{5} - 2\right| = \left|\frac{4x-12}{5}\right| = \frac{4}{5}|x-3|.$$
Thus given any $\varepsilon > 0$, we can take $\delta = \frac{5}{4}\varepsilon$ and then for any x such that $0 < |x-3| < \delta$, we have
$$\left|\frac{4x-2}{5} - 2\right| = \frac{4}{5}|x-3| < \frac{4}{5}\delta = \frac{4}{5}\cdot\frac{5}{4}\varepsilon = \varepsilon.$$

Question 2.

(a) State Rolle's Theorem.

(b) Use Rolle's Theorem to prove that the polynomial $f(x) = x^2$ has precisely one real root. [Hint: Assume for contradiction that $f(a) = 0$ for some $a \neq 0$.]

(c) Consider the function $f(x) = |2x-1|$. Note that $f(-2) = f(3)$, but there is no $x \in (-2, 3)$ satisfying $f'(x) = 0$. Does this contradict Rolle's Theorem? (Provide an explanation for your YES/NO answer.)

Solution.

(a) Let f be a function that is continuous in an interval $[a, b]$ and differentiable in (a, b). If $f(a) = f(b)$, then there is $c \in (a, b)$ such that $f'(c) = 0$.

(b) Assume that there is a number $a \neq 0$ such that $f(a) = 0$. Since f is differentiable everywhere, Rolle's Theorem applies over any interval. In particular, if $a > 0$ there is $c \in (0, a)$ such that $f'(c) = 0$. (Similarly, if $a < 0$, there is $c \in (a, 0)$ such that $f'(c) = 0$.) In any case, there is $c \neq 0$ such that $f'(c) = 0$. However, $f'(x) = 2x$ so if $f'(c) = 0$, then $c = 0$ which is a contradiction.

(c) This does NOT contradict Rolle's Theorem. In order to apply the theorem, we would need to know that f is differentiable everywhere in $(-2, 3)$. That is not true. Specifically, f is not differentiable at $\frac{1}{2}$. Indeed, we have
$$f(x) = \begin{cases} 2x-1 & \text{if } x \geq \frac{1}{2} \\ -2x+1 & \text{if } x < \frac{1}{2}. \end{cases}$$

Therefore,

$$\lim_{x\to\frac{1}{2}^+}\frac{f(x)-f(\frac{1}{2})}{x-\frac{1}{2}} = \lim_{x\to\frac{1}{2}^+}\frac{2x-1}{x-\frac{1}{2}} = 2$$
$$\text{and } \lim_{x\to\frac{1}{2}^-}\frac{f(x)-f(\frac{1}{2})}{x-\frac{1}{2}} = \lim_{x\to\frac{1}{2}^-}\frac{-2x+1}{x-\frac{1}{2}} = -2.$$

Thus the limit $\lim_{x\to\frac{1}{2}}\frac{f(x)-f(\frac{1}{2})}{x-\frac{1}{2}}$ does not exist, i.e. $f'(\frac{1}{2})$ does not exist.

Question 3. Evaluate $\int \sqrt{e^x}\sin x\,\mathrm{d}x$.

Solution. Define

$$A(x) = \int \sqrt{e^x}\sin x\,\mathrm{d}x = \int e^{\frac{x}{2}}\sin x\,\mathrm{d}x \text{ and } B(x) = \int e^{\frac{x}{2}}\cos x\,\mathrm{d}x.$$

Compute $A(x)$ using integration by parts with $u = \sin x$ and $v = 2e^{\frac{x}{2}}$ so that $\mathrm{d}u = \cos x\,\mathrm{d}x$ and $\mathrm{d}v = e^{\frac{x}{2}}\,\mathrm{d}x$.

$$A(x) = \int e^{\frac{x}{2}}\sin x\,\mathrm{d}x = 2e^{\frac{x}{2}}\sin x - \int 2e^{\frac{x}{2}}\cos x\,\mathrm{d}x = 2e^{\frac{x}{2}}\sin x - 2B(x)$$

Similarly, compute $B(x)$ with $u = \cos x$ and $v = 2e^{\frac{x}{2}}$ so that $\mathrm{d}u = -\sin x\,\mathrm{d}x$ and $\mathrm{d}v = e^{\frac{x}{2}}\,\mathrm{d}x$.

$$B(x) = \int e^{\frac{x}{2}}\cos x\,\mathrm{d}x = 2e^{\frac{x}{2}}\cos x + \int 2e^{\frac{x}{2}}\sin x\,\mathrm{d}x = 2e^{\frac{x}{2}}\cos x + 2A(x)$$

Altogether, we obtain

$$A(x) = 2e^{\frac{x}{2}}\sin x - 2B(x) = 2e^{\frac{x}{2}}\sin x - 2(2e^{\frac{x}{2}}\cos x + 2A(x)) = 2e^{\frac{x}{2}}\sin x - 4e^{\frac{x}{2}}\cos x - 4A(x).$$

Solving for $A(x)$ yields

$$A(x) = \frac{1}{5}e^{\frac{x}{2}}(2\sin x - 4\cos x) + C.$$

Question 4. Evaluate $\int \cos t\cos^3(\sin t)\,\mathrm{d}t$.

Solution. Substituting $u = \sin t$ so that $\mathrm{d}u = \cos t\,\mathrm{d}t$, we get

$$\int \cos t\cos^3(\sin t)\,\mathrm{d}t = \int cos^3 u\,\mathrm{d}u = \int \cos u(1-\sin^2 u)\,\mathrm{d}u.$$

Next, we substitute $v = \sin u$ so that $\mathrm{d}v = \cos u\,\mathrm{d}u$ and continue to get

$$\int \cos u(1-\sin^2 u)\,\mathrm{d}u = \int 1-v^2\,\mathrm{d}v = v-\frac{v^3}{3}+C = \sin u-\frac{1}{3}\sin^3 u+C = \sin(\sin t)-\frac{1}{3}\sin^3(\sin t)+C.$$

Question 5. Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{4-x^2}}\,\mathrm{d}x$.

You may ignore this question. It comes from Section 7.3 which is not included in the Winter 2016 course.

Question 6.

(a) Write out the form of the partial fraction decomposition for the function

$$\frac{x^5 - 2x^4 - 7x + 2}{(x^2+3x+3)(x^2-2x-3)(x^2-1)}.$$

Do NOT evaluate the coefficients.

(b) Evaluate $\displaystyle\int \frac{3x^2+x}{x^2+x-2}\,\mathrm{d}x$.

Solution.

(a) Let $f(x)$ and $g(x)$ denote the numerator and the denominator of this expression. We need to decompose g into irreducible factors.

The discriminant of x^2+3x+3 is $3^2 - 4\cdot 3 = -3$ so this quadratic polynomial is irreducible. However, we have decompositions $x^2-2x-3 = (x-3)(x+1)$ and $x^2-1 = (x+1)(x-1)$. Therefore,

$$g(x) = (x^2+3x+3)(x-3)(x-1)(x+1)^2$$

and (since $\deg f < \deg g$) the partial fraction decomposition has the form

$$\frac{f(x)}{g(x)} = \frac{Ax+B}{x^2+3x+3} + \frac{C}{x-3} + \frac{D}{x-1} + \frac{E}{x+1} + \frac{F}{(x+1)^2}.$$

(b) First, we observe that $3x^2+x = 3(x^2+x-2) - 2x + 6$ so that

$$\int \frac{3x^2+x}{x^2+x-2}\,\mathrm{d}x = \int 3\,\mathrm{d}x + \int \frac{-2x+6}{x^2+x-2}\,\mathrm{d}x.$$

Next, we factor $x^2+x-2 = (x+2)(x-1)$ and obtain a partial fraction decomposition

$$\frac{-2x+6}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1},$$
$$\text{i.e.} \quad -2x+6 = A(x-1) + B(x+2).$$

By taking $x=1$ in the latter equation, we obtain $4 = B\cdot 3$ so $B = \frac{4}{2}$. By taking $x=-2$, we get $10 = A\cdot(-3)$ so $A = -\frac{10}{3}$. Therefore,

$$\int \frac{3x^2+x}{x^2+x-2}\,\mathrm{d}x = \int 3\,\mathrm{d}x - \frac{10}{3}\int \frac{1}{x+2}\,\mathrm{d}x + \frac{4}{3}\int \frac{1}{x-1}\,\mathrm{d}x = 3x - \frac{10}{3}\ln|x+2| + \frac{4}{3}\ln|x-1| + C.$$

Question 7. Determine whether $\displaystyle\int_0^\infty \frac{1}{(x+1)^{\frac{3}{2}}}\,\mathrm{d}x$ is convergent. If so, compute its precise value.

Solution. For every $t > 0$, the function $\frac{1}{(x+1)^{\frac{3}{2}}}$ is well-defined and continuous everywhere in the interval $[0, t]$. Therefore, we can compute its integral from 0 to t using the Fundamental Theorem of Calculus:

$$\int_0^t \frac{1}{(x+1)^{\frac{3}{2}}} \, \mathrm{d}x = \left[\frac{-2}{\sqrt{x+1}}\right]_0^t = \frac{-2}{\sqrt{t+1}} - \frac{-2}{\sqrt{1}} = 2 - \frac{2}{\sqrt{t+1}}.$$

The limit of this expression as $t \to \infty$ is 2 and thus

$$\int_0^\infty \frac{1}{(x+1)^{\frac{3}{2}}} \, \mathrm{d}x = 2.$$

Question 8. Determine whether $\displaystyle\int_{-\infty}^{-1} e^{-2x} \, \mathrm{d}x$ is convergent. If so, compute its precise value.

Solution. As in the previous question, for any $t < -1$ we can compute

$$\int_t^{-1} e^{-2x} \, \mathrm{d}x = \left[-\frac{1}{2} e^{-2x}\right]_t^{-1} = -\frac{1}{2} e^2 + \frac{1}{2} e^{-2t}$$

which tends to ∞ as $t \to -\infty$. Thus the integral $\displaystyle\int_{-\infty}^{-1} e^{-2x} \, \mathrm{d}x$ diverges.

Question 9. Use the Comparison Theorem to determine the convergence of $\displaystyle\int_2^\infty \frac{\ln(x-1)}{x^3 + \sqrt{x}} \, \mathrm{d}x$. Do NOT compute the precise value of the integral.

Solution. For all $x \geq 2$ we have

$$\begin{aligned} f(x) = \frac{\ln(x-1)}{x^3 + \sqrt{x}} &\geq 0, \\ \ln(x-1) &\leq x, \\ x^3 + \sqrt{x} &\geq x^3. \end{aligned}$$

Thus $f(x) \leq \frac{x}{x^3} = \frac{1}{x^2}$ (and both these functions are continuous in $[2, \infty)$). The integral $\displaystyle\int_2^\infty \frac{1}{x^2} \, \mathrm{d}x$ is convergent and therefore so is $\displaystyle\int_2^\infty \frac{\ln(x-1)}{x^3 + \sqrt{x}} \, \mathrm{d}x$ by the Comparison Theorem.

Question 10. Is the following statement TRUE or FALSE:

> "Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$. If $\int_a^\infty f(x) \, \mathrm{d}x$ is divergent, then $\int_a^\infty g(x) \, \mathrm{d}x$ must be divergent as well."

If TRUE, provide an argument or explanation. If FALSE, provide a counterexample.

Solution. The statement is FALSE. (In particular, it is not the statement of the Comparison Theorem. The conclusion of the Comparison Theorem would be "if $\int_a^\infty f(x) \, \mathrm{d}x$ is convergent, then $\int_a^\infty g(x) \, \mathrm{d}x$ must be convergent as well" or "if $\int_a^\infty g(x) \, \mathrm{d}x$ is divergent, then $\int_a^\infty f(x) \, \mathrm{d}x$ must be divergent as well".)

For a counterexample, consider $a = 1$, $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$. Then for each $x \geq a$, $f(x) \geq g(x) \geq 0$ while $\int_a^\infty f(x) \, \mathrm{d}x$ is divergent and $\int_a^\infty g(x) \, \mathrm{d}x$ is convergent.