2015 Midterm 2 solutions

1. $\boxed{B}$ $\quad 0 \le \left| \frac{2n\sin(1-n)}{3n^2+1} \right| \overset{\text{for all } n}{\le} \frac{2n}{3n^2+1} < \frac{2n}{3n^2} = \frac{2}{3n} \qquad \lim_{n\to\infty} \frac{2}{3n} = 0$

$$\underset{\text{Squeeze Theorem}}{\Longrightarrow} \lim_{n\to\infty} \left| \frac{2n\sin(1-n)}{3n^2+1} \right| = 0 \Rightarrow \lim_{n\to\infty} \frac{2n\sin(1-n)}{3n^2+1} = 0$$

2. $\boxed{C}$ Disregard this question

3. $\boxed{D}$ Notice that $\lim_{n\to\infty} \frac{\ln(\ln n)}{n} = \lim_{x\to\infty} \frac{\ln(\ln x)}{x} \underset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{(\ln(\ln x))'}{x'}$

$$= \lim_{x\to\infty} \frac{\frac{1}{\ln x}\frac{1}{x}}{1} = 0$$

therefore

$$\lim_{n\to\infty} \frac{(n-\ln(\ln n))(n+\ln(\ln n))}{3n^2+1} = \lim_{n\to\infty} \frac{\frac{n-\ln(\ln n)}{n} \cdot \frac{n+\ln(\ln n)}{n}}{\frac{3n^2+1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\left(1-\frac{\ln(\ln n)}{n}\right)\left(1+\frac{\ln(\ln n)}{n}\right)}{3+\frac{1}{n^2}} = \frac{(1-0)(1+0)}{3+0} = \frac{1}{3}$$

4. Apply Alternating Series Estimation Theorem

$$|S-S_3| = |R_3| \le |b_4| = \frac{2^4}{4^4 4^4} = \frac{1}{8^4} \approx 0.00024 < 0.001$$

therefore <u>three</u> terms will be sufficient

$|b_3| = \frac{2^3}{4^3 3^4} \approx 0.0015$, so we can not conclude that 2 terms would be enough

5. $\boxed{D}$ $\quad \lim_{n\to\infty} \frac{\Gamma(n+1)\Gamma(n+2)}{(\Gamma(n))^2(2n-1)^3} = \lim_{n\to\infty} \frac{n!\,(n+1)!}{(n-1)!\,(n-1)!\,(2n-1)^3} = \lim_{n\to\infty} \frac{n(n+1)n}{(2n-1)^3} = \frac{1}{8}$

6. ADFH

7. CFH

8. (a) $\lim_{n\to\infty} a_n = L$ if for any $\varepsilon > 0$ there is N such that

$|a_n - L| < \varepsilon$ for all $n > N$

(b) $\left|\frac{2n+1}{5n} - \frac{2}{5}\right| < \varepsilon \iff \left|\frac{2n+1-2n}{5n}\right| < \varepsilon \iff \frac{1}{5n} < \varepsilon \iff n > \frac{1}{5\varepsilon}$ (*)

Take any $\varepsilon > 0$. Choose N (sufficiently large) such that $N > \frac{1}{5\varepsilon}$.

Then for all $n > N$

$$n > N > \frac{1}{5\varepsilon} \underset{(*)}{\Rightarrow} \left|\frac{2n+1}{5n} - \frac{2}{5}\right| < \varepsilon$$

Therefore $\lim_{n\to\infty} \frac{2n+1}{5n} = \frac{2}{5}$.

9. (a) A bounded monotonic sequence is convergent.

(b) <u>Proof:</u>

$\{S_n\}$ is increasing (because $S_{n+1} - S_n = a_{n+1} > 0$ for all n; $S_{n+1} > S_n$)

$\{S_n\}$ is bounded $\left(\begin{array}{l} S_n > 0 \text{ because all } a_i > 0 \quad i = 1,2,3,\ldots \\ S_n \le 2015 + \cos(\pi n^{2015}) \le 2015 + 1 = 2016 \end{array} \text{ for all } n\right)$

$\underset{MCT}{\Rightarrow}$ $\{S_n\}$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges $\square$

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Yes, the series is absolutely convergent, because $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n$ converges.

10. $\boxed{\text{Answer: } I = \{-1\}}$

Explanation: Ratio Test $a_n = \frac{2^{n^2}}{n!}(x+1)^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{(n+1)^2}}{(n+1)!}(x+1)^{n+1} \div \frac{2^{n^2}}{n!}(x+1)^n\right| = \lim_{n\to\infty} \frac{2^{(n+1)^2 - n^2}}{n+1}|x+1|$$

$$= \lim_{n\to\infty} \frac{2^{2n+1}}{n+1}|x+1| = \begin{cases} \infty & \text{if } x \ne -1 \text{ the series diverges} \\ 0 & \text{if } x = -1 \text{ the series converges} \end{cases}$$

11. (a) Suppose $f(x)$ is a continuous, positive, decreasing function on $[1,\infty)$, $a_n = f(n)$ $n=1,2,3,\ldots$

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges}$$

(b) Let $f(x) = xe^{-x^2}$ $\quad f(n) = ne^{-n^2}$ $\quad n=1,2,3,\ldots$

$f(x)$ is continuous and positive on $[1,\infty)$

$$f'(x) = e^{-x^2} + xe^{-x^2}(-2x) = e^{-x^2}(1-2x^2) < 0 \quad \text{on } [1,\infty)$$

$f(x)$ is decreasing

$$\int_1^{\infty} xe^{-x^2}dx = \lim_{t\to\infty}\int_1^t xe^{-x^2}dx = \lim_{t\to\infty}\int_{-1}^{-t^2} e^u\left(-\tfrac{1}{2}du\right) = \lim_{t\to\infty}\left(-\tfrac{1}{2}\right)\left(e^{-t^2}-e^{-1}\right)$$

$u=-x^2$

$du=-2x\,dx$

$$= -\tfrac{1}{2}\left(0-\tfrac{1}{e}\right) \text{ convergent}$$

$$\underset{IT}{\Rightarrow} \sum_{n=1}^{\infty} ne^{-n^2} \text{ converges}$$

12. $$\frac{\Gamma(c+1)\Gamma(c+2)}{c^2(c+1)} = \frac{c\Gamma(c)(c+1)\Gamma(c+1)}{c^2(c+1)} = \frac{c\Gamma(c)(c+1)c\Gamma(c)}{c^2(c+1)} = (\Gamma(c))^2 = \boxed{\frac{\pi^2}{4}}$$