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CALC 1501 LECTURE NOTES

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1. Mean Value Theorem

1.1. **Review: limit, continuity, differentiability.** We denote by $\mathbb{R}$ the set of real numbers. A *domain* D of $\mathbb{R}$ is any subset of $\mathbb{R}$. Typically this will be on *open* interval (a, b) or a *closed* interval $[a, b]$. A function of a real variable is a function $f : D \to \mathbb{R}$, where D is a domain of $\mathbb{R}$.

Definition 1.1 (The $\epsilon - \delta$ Definition)**.** *We say that a function $f(x)$ has a limit L as x approaches a point x_0 and write $\lim_{x\to x_0} f(x) = L$, if for any $\epsilon > 0$ there exists $\delta > 0$ such that whenever $0 < |x - x_0| < \delta$ (and $x \in D$) we have $|f(x) - L| < \epsilon$.*

The meaning of the above definition is that by choosing a sufficiently small interval $(x_0-\delta, x_0+\delta)$ of the point x_0 we can ensure that the values of $f(x)$ on this interval (excluding x_0) do not deviate from L by more than ϵ.

Example 1.1. We will use this definition to prove that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. For this we need to show that for any $\epsilon > 0$, there exists a choice of $\delta > 0$ such that

$$\left|\frac{\sin x}{x} - 1\right| < \epsilon, \quad \text{whenever} \quad |x| < \delta.$$

First we recall the following inequality from trigonometry: for $0 < x < \pi/2$,

$$\sin x < x < \tan x. \tag{1.1}$$

If we divide $\sin x$ by the three terms in the above inequality we obtain

$$\frac{\sin x}{\sin x} > \frac{\sin x}{x} > \frac{\sin x}{\tan x} \quad \Rightarrow \quad 1 > \frac{\sin x}{x} > \cos x.$$

From this we conclude that

$$0 < 1 - \frac{\sin x}{x} < 1 - \cos x = 2\sin^2\frac{x}{2} < 2\sin\frac{x}{2} < x,$$

where in the last step we again used inequality (1.1). It follows that

$$\left|\frac{\sin x}{x} - 1\right| < |x|.$$

The same inequality holds for $x < 0$ because $\frac{\sin(-x)}{-x} = \frac{\sin x}{x}$. So, given $\epsilon > 0$, we can take $\delta = \min\{\epsilon, \pi/2\}$ to satisfy the definition of limit. $\diamond$

Definition 1.2. *We say that a function $f : D \to \mathbb{R}$ is continuous at a point $x_0 \in D$ if*

$$\lim_{x\to x_0} f(x) = f(x_0). \tag{1.2}$$

Using the $\epsilon - \delta$ definition this can be stated as follows: given $\epsilon > 0$, there exists $\delta > 0$ such that whenever $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \epsilon$.

Example 1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x$. Let x_0 be any real number. Then $f(x)$ is continuous at x_0. Indeed, using the $\epsilon - \delta$ definition we have $|f(x) - f(x_0)| = |x - x_0| < \epsilon$. This inequality can be ensured by taking $\delta = \epsilon$. $\diamond$

Theorem 1.3. *If f and g are continuous functions on a domain D, then so are the functions $f+g$, $f \cdot g$, and $c \cdot f$, where c is any constant. The function f/g is continuous at all points of D where $g \neq 0$. Further, if g is a function defined on the range of f, then the function $g \circ f = g(f(x))$ is continuous on D.*

Using the above theorem and the fact that $f(x) = x$ is a continuous function as shown in Example 1.2, we conclude that any polynomial is a continuous function, and any rational function (the quotient of two polynomials) is continuous at all points where the denominator does not vanish.

Example 1.3. Let $f(x) = \sqrt{x}$. We will use the $\epsilon - \delta$ definition to show that this function is continuous at any point $x_0 > 0$. Observer that

$$|\sqrt{x} - \sqrt{x_0}| = \frac{(|\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0})|}{\sqrt{x} + \sqrt{x_0}} = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} < \frac{|x - x_0|}{\sqrt{x_0}}$$

Now, let $\epsilon > 0$ be arbitrary. We choose $\delta = \epsilon\sqrt{x_0}$ (x_0 is a fixed number!). Then for $|x - x_0| < \delta$, we have

$$|\sqrt{x} - \sqrt{x_0}| < \frac{|x - x_0|}{\sqrt{x_0}} < \frac{\epsilon\sqrt{x_0}}{\sqrt{x_0}} = \epsilon,$$

which proves the continuity. $\diamond$

Example 1.4. Let

$$f(x) = \begin{cases} 0, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Then $\lim_{x\to 0} f(x)$ exists and equals zero, but it differs from the value of f at the origin since $f(0) = 1$. Therefore, equation (1.2) does not hold, and $f(x)$ is not continuous at the origin. However, letting $f(0) = 0$ will make this function continuous everywhere. $\diamond$

Example 1.5 (Dirichlet's function)**.** Recall that a *rational number* is the quotient of two integers. The set of all rational numbers is denoted by $\mathbb{Q}$. All real numbers that are not rational are called *irrational*. They form a set $\mathbb{R} \setminus \mathbb{Q}$. Define

$$d(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

This function is discontinuous at all points. Indeed, let x_0 be any real number, say x_0 is rational. Then for any $\delta > 0$, the interval $(x_0 - \delta, x_0 + \delta)$ necessarily contains an irrational number x, and then, $|d(x) - d(x_0)| = 1$. Thus, for $\epsilon < 1$, no choice of δ will satisfy the condition of Definition 1.2. A similar argument will work if x_0 is irrational. $\diamond$

Example 1.6. Let

$$f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{1.3}$$

The function f (see Fig. 1) is defined for all x. It is continuous for all $x \neq 0$ because it is a composition of a continuous functions $1/x$ and $\sin x$. But $f(x)$ does not have a limit as $x \to 0$ (why?), and therefore $f(x)$ is not continuous at the origin. Note that there is no choice of $f(0)$ that will make this function continuous at the origin. $\diamond$

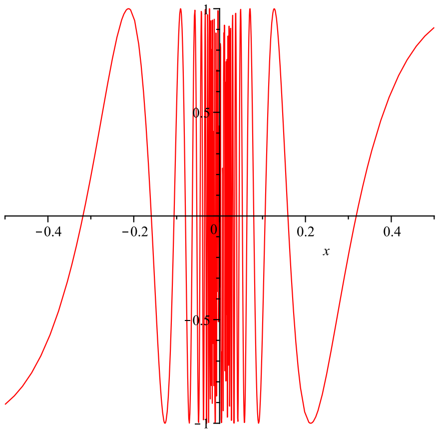

FIGURE 1. The graph of $\sin \frac{1}{x}$

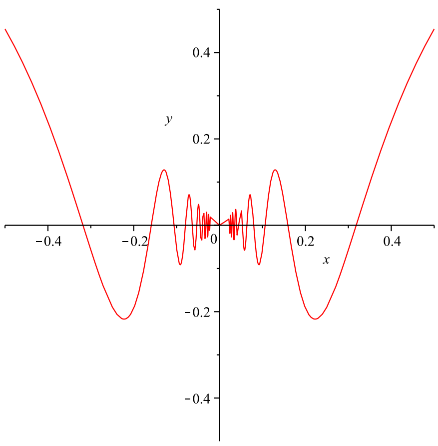

FIGURE 2. The graph of $x \sin \frac{1}{x}$

Example 1.7. Let

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

This function (see Fig. 2) is continuous everywhere. To prove the continuity at the origin, let us verify the $\epsilon - \delta$ definition. We have

$$|f(x) - f(0)| = \left| x \sin \frac{1}{x} \right| < \epsilon.$$

Since $|x \sin \frac{1}{x}| < |x|$ for all $x \neq 0$, we have

$$|f(x) - f(0)| = \left| x \sin \frac{1}{x} \right| \leq |x| < \epsilon,$$

and so we may take $\delta = \epsilon$. Intuitively, $\lim_{x \to 0} x \sin \dfrac{1}{x} = 0$ because $\sin \frac{1}{x}$ is bounded between -1 and 1, whereas x approaches zero. $\diamond$

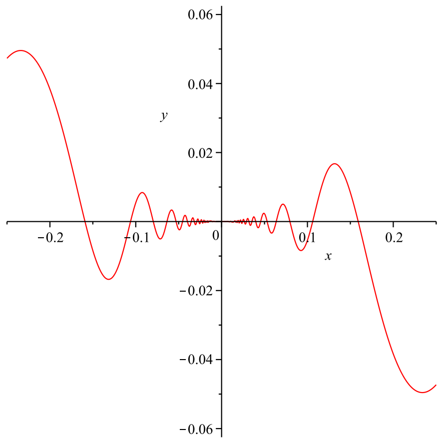

FIGURE 3. The graph of $x^2 \sin \frac{1}{x}$

Definition 1.4. *Let $f(x)$ be defined on an interval $D \subset \mathbb{R}$. Let $x_0 \in D$. We say that $f(x)$ is differentiable at x_0 if the limit*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{1.4}$$

exists. The value of the limit is defined to be $f'(x_0)$, the derivative of f at x_0.

Example 1.8. Let us apply the above definition to the function $f(x) = x^2$. The expression under the limit in equation (1.4) becomes

$$\frac{(x_0 + h)^2 - x_0^2}{h} = \frac{x_0^2 + 2x_0 h + h^2 - x_0^2}{h} = 2x_0 + h.$$

Clearly, the limit of the above expression equals $2x_0$, as $h \to 0$. Thus, we proved that $f(x) = x^2$ is differentiable at every point, and $(x^2)' = 2x$. $\diamond$

Example 1.9. Let

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

This function is continuous but not differentiable at the origin. The continuity was shown in Example 1.7. As for nondifferentiability, we have

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h \sin \frac{1}{h}}{h} = \lim_{h \to 0} \sin \frac{1}{h},$$

which does not exist. $\diamond$

Example 1.10. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

This function (see Fig. 3) is continuous everywhere because it is the product of continuous functions x and $x \sin 1/x$ (as discussed in Example 1.7). To prove differentiability of this function at the origin let us compute the corresponding limit in (1.4).

$$\lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h}.$$

As we saw in Example 1.7 this limit equals 0. Thus $f'(0) = 0$. $\diamond$

1.2. **Mean Value Theorem.**

Definition 1.5. *Suppose $f(x)$ is a function defined on a domain D. The function $f(x)$ is said to have an absolute (global) maximum at a point $c \in D$, if $f(c) \geq f(x)$ for all $x \in D$. The number $f(c)$ is called the absolute (global) maximum value of f on the domain D. The function f has an absolute (global) minimum at $c \in D$, if $f(c) \leq f(x)$ for all $x \in D$. The number $f(c)$ is called the absolute (global) minimum value of f on the domain D.*

Example 1.11. Consider a constant function $f(x) = c$, for some $c \in \mathbb{R}$. Then every point x is a global maximum and minimum of $f(x)$. On the other hand, the function $f(x) = x^3$ for $x \in \mathbb{R}$ does not attain a global maximum or minimum. The same is true if we consider this function on any *open* interval (a, b).

Theorem 1.6. *If $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ attains a maximum and a minimum value.*

The above theorem can be proved using the Axiom of Completeness for real numbers which will be stated when we discuss sequences.

Definition 1.7. *The function f defined on a domain D has a local maximum at a point $c \in D$, if there is an open interval $I \subset D$, such that $c \in I$, and $f(c) \geq f(x)$ for all $x \in I$. The function f has a local minimum at $c \in D$, if there is an open interval $I \subset D$, such that $c \in I$, and $f(c) \leq f(x)$ for all $x \in I$.*

Maxima and minima are called extreme points, or extrema.

Lemma 1.8. *Let $f(x)$ be a differentiable function on an interval (a, b). Suppose $x_0 \in (a, b)$. If $f'(x_0) > 0$, then for $x < x_0$ close to x_0 we have $f(x) < f(x_0)$, and $f(x) > f(x_0)$ for $x > x_0$ and close to x_0.*

The lemma above simply states that if $f'(x_0) > 0$, then $f(x)$ is an increasing function near x_0. A similar statement holds if we assume that $f'(x_0) < 0$ (see Exercise 1.6).

Proof. By definition,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

If $f'(x_0) > 0$, then there exists a small interval $(x_0 - \delta, x_0 + \delta)$ such that

$$\frac{f(x) - f(x_0)}{x - x_0} > 0, \quad \text{for } x \neq x_0.$$

Suppose first that $x_0 < x < x_0 + \delta$. Then $x - x_0 > 0$, and from the above inequality we conclude that $f(x) - f(x_0) > 0$, or $f(x) > f(x_0)$. Now, if $x_0 - \delta < x < x_0$, then $x - x_0 < 0$, and the same inequality shows that $f(x) < f(x_0)$. $\square$

Theorem 1.9 (Fermat's Theorem). [1] *Let $f(x)$ be defined on an interval $[a, b]$, and suppose that $f(x)$ attains a maximal (or minimal) value at a point $c \in (a, b)$. If $f(x)$ is differentiable at $x = c$, then $f'(c) = 0$.*

[1]This is a modern formulation of the theorem. It captures the essence of Fermat's method for finding maximal and minimal values of a function. The notion of derivative was not yet developed at Fermat's time.

Proof. We will assume that c is a maximum of $f(x)$, the case when c is a minimum can be treated in a similar way. Arguing by contradiction, suppose that $f'(c) \neq 0$. Then either $f'(c) > 0$ or $f'(c) < 0$. If $f'(c) > 0$, then Lemma 1.8 implies that $f(x) > f(c)$ for $x > c$ with x sufficiently close to c. Similarly, if $f'(c) < 0$, then $f(x) > f(c)$ for $x < c$. In both cases we see that $f(c)$ cannot be the maximum value of the function f. This contradiction proves the theorem. □

Geometrically, Fermat's theorem states that at extreme points the tangent line to the graph of the function f is horizontal, which should be intuitively clear. Also note, that if a maximal or a minimum value is attained at the end point of the interval $[a, b]$, then Fermat's theorem need not to hold.

Definition 1.10. *A point c is called a critical point of a differentiable function $f(x)$ if $f'(c) = 0$.*

Fermat's theorem now can be stated as follows: if c is a local maximum or minimum of a function $f(x)$, then c is a critical point of f. The converse to this statement is false: if $f'(c) = 0$, then it does not follow in general that c is a local maximum or a local minimum of $f(x)$. For example, if $f(x) = x^3$, then $f'(0) = 0$, but the origin is not an extreme point of x^3.

Theorem 1.11 (Rolle's Theorem)**.** [2] *Suppose $f(x)$ is continuous on the interval $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Then there exists a number $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. By Theorem 1.6, a continuous function on a closed interval $[a, b]$ attains its maximum value, say, M, and its minimum value, say, m. Consider two cases:

1. Suppose $M = m$. Then $f(x)$ on $[a, b]$ is a constant function, since $m \le f(x) \le M = m$ for all $x \in [a, b]$. Therefore, $f'(x) = 0$ for all x.

2. Suppose $M > m$. Since $f(a) = f(b)$, we know that either M or m is attained at some point c inside the interval (a, b), (i.e., not at the end points of the interval). In this case, it follows from Fermat's theorem that $f'(c)$ must be zero. □

Geometrically, Rolle's theorem states that if $f(a) = f(b)$, then there is a point c between a and b such that the tangent line to the graph of f at point c is horizontal. This occurs at a local maximum or a local minimum of $f(x)$.

Theorem 1.12 (Mean Value Theorem)**.** *Suppose that $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Then there exists a point $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof. Define an auxiliary function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

This function satisfies the conditions of Rolle's theorem. Indeed, it is continuous on $[a, b]$, because it is a difference of a continuous function $f(x)$ and a linear (hence continuous!) function

$$f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

On the interval (a, b), we have

$$F'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

[2]Despite the name, Michel Rolle only suggested this result for polynomials in 1691.

Finally, $F(a) = f(a) - f(a) = 0$, and $F(b) = f(b) - f(a) - \frac{f(b)-f(a)}{b-a}(b-a) = f(b) - f(a) - (f(b) - f(a)) = 0$, and so $F(a) = F(b)$.

Therefore, we may apply Rolle's theorem to the function $F(x)$, and so there exists a point $c \in (a, b)$ such that $F'(c) = 0$. This means that

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0.$$

Hence,

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

which is exactly what we wanted to prove. □

Using the Mean Value Theorem we can now prove that only constant functions have everywhere vanishing derivatives.

Corollary 1.13. *Suppose $f(x)$ is a differentiable functions such that $f'(x) = 0$ for all x. Then $f(x)$ is a constant function.*

Proof. Choose any two points a and b in the domain of $f(x)$, say, $a < b$. By the Mean Value Theorem, there exists a point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) = 0.$$

It follows then that $f(b) = f(a)$. But this means that $f(x)$ is a constant function. □

1.3. **Proving inequalities.** The Mean Value Theorem can be used for proving inequalities.

Example 1.12. Prove that if $x > 0$, then

$$\ln(1 + x) < x.$$

Solution. Let $a = 0$, $b = x$, and $f(x) = \ln(1 + x) - x$. Then $f'(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x}$. By the Mean Value Theorem applied to the function f on the interval $[a, b] = [0, x]$, there exists a point $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0},$$

or

$$-\frac{c}{1 + c} = \frac{\ln(1 + x) - x}{x}. \tag{1.5}$$

Note that $c > 0$, and therefore, $-\frac{c}{1+c} < 0$. Therefore, equation (1.5) implies

$$\frac{\ln(1 + x) - x}{x} < 0.$$

Since $x > 0$, the numerator in the above inequality must be negative, i.e.,

$$\ln(1 + x) - x < 0,$$

which is what we had to prove. ⋄

Example 1.13. Prove that if $x > 0$, and $n > 1$, then

$$(1 + x)^n > 1 + nx.$$

Solution. Let $a = 0$, and $b = x$, and $f(x) = (1 + x)^n - (1 + nx)$. Then $f'(x) = n(1 + x)^{n-1} - n$, and by the Mean Value Theorem, we have

$$n(1 + c)^{n-1} - n = \frac{(1 + x)^n - (1 + nx) - 0}{x} \tag{1.6}$$

for some $c \in (0, x)$. Note that $1 + c > 1$, and for $n > 1$, we have $(1 + c)^{n-1} > 1$. Therefore,

$$n(1 + c)^{n-1} - n > 0.$$

From this and equation (1.6) we conclude that

$$\frac{(1 + x)^n - (1 + nx)}{x} > 0.$$

Since $x > 0$, this yields the desired inequality. ⋄

Exercises

1·1. Use a similar strategy as in Example 1.3 to show that the following functions are continuous on the specified domain:
  (a) $f(x) = 2x + 1$, for $x_0 \in \mathbb{R}$,
  (b) $f(x) = x^2$, for $x_0 \in \mathbb{R}$,
  (c) $f(x) = 1/x$ for $x_0 \neq 0$.

1·2. Prove, using the definition, that the function $f(x) = x^3$ is differentiable at all points.

1·3. Negate the $\epsilon - \delta$ definition of the limit to write what it means that a function $f(x)$ defined for $x \neq 0$ does not have a limit as x approaches the origin. Use this to prove that the function

$$f(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

does not have a limit at zero.

1·4. Show that the function given by (1.3) is not continuous at the origin.

1·5. Show that the function in Example 1.10 does not have the second order derivative at $x = 0$.

1·6. Formulate and prove a statement similar to Lemma 1.8 for the case when $f'(x_0) < 0$.

1·7. Give an example of a function which is defined on the closed interval $[0, 1]$ but is not bounded there.

1·8. Give an example of a function which is continuous on the interval $(-\infty, 0]$ but does not attain a global maximum and a global minimum.

1·9. Prove that if a polynomial $p(x)$ vanishes at two points a and b, then there exists a point c between a and b such that $p'(c) = 0$.

1·10. Prove that if a polynomial $p(x)$ of degree 3 has 3 pairwise different (real) roots, then $p'(x)$ has exactly two (real) roots.

1·11. On the interval $(0, 1)$ find a point c such that the tangent line to the graph of the function $y = x^3$ at the point (c, c^3) is parallel to the straight line passing through the points $(0, 0)$ and $(1, 1)$.

1·12. Prove that if a nonconstant function $f(x)$ satisfies the conditions of Rolle's theorem on the interval $[a, b]$, then there exist points x_1 and x_2 on the interval (a, b) such that $f'(x_1) < 0$ and $f'(x_2) > 0$.

1·13. Prove that if $f(x)$ is a 3 times differentiable function on $x \geq 0$, and $f(0) = f'(0) = f''(0) = 0$, and $f'''(x) > 0$ for $x > 0$, then also $f(x) > 0$ for $x > 0$.

1·14. (Cauchy's Theorem) If the functions $x = \phi(t)$ and $y = \psi(t)$ are continuous on the interval $[a, b]$ and differentiable on (a, b) with $\phi'(t) \neq 0$ for $a < t < b$, then there exists a point $\xi \in (a, b)$ such that

$$\frac{\psi(b) - \psi(a)}{\phi(b) - \phi(a)} = \frac{\psi'(\xi)}{\phi'(\xi)}. \tag{1.7}$$

Hint: Consider an auxiliary function $h(x) = (\psi(b) - \psi(a))\phi(x) - (\phi(b) - \phi(a))\psi(x)$.

1·15. Suppose that $f(x)$ is a continuous function on $[0, \infty)$, differentiable on $(0, \infty)$, $f(0) = 0$, and $f'(x)$ is an increasing function for $x > 0$. Prove that the function $\dfrac{f(x)}{x}$ is also increasing for $x > 0$.

In the next problems prove the given inequality using the Mean Value Theorem.

1·16. $2\sqrt{x} > 3 - \dfrac{1}{x}$, for $x > 1$.

1·17. $\sin x < x$, for $x > 0$.

1·18. $\cos x > 1 - \dfrac{x^2}{2}$, for $x > 0$.

1·19. $\sin x > x - \dfrac{x^3}{6}$, for $x > 0$.

1·20. $\tan x > x$, for $0 < x < \frac{\pi}{2}$.

1·21. $e^x > 1 + x$, for $x > 0$.

1·22. $e^x > 1 + x + \dfrac{x^2}{2}$, for $x > 0$.

1·23. $e^x > 1 + x + \dfrac{x^2}{2} + \cdots + \dfrac{x^n}{n!}$, for $x > 0$. (*Hint:* use the mathematical induction)