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CALC 1501 LECTURE NOTES

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5. Series

5.1. **Basic Definitions.** Given a sequence of real numbers

$$a_1, a_2, a_3, \dots, a_n, \dots$$

a formal expression

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n \tag{5.1}$$

is called an *infinite series*, or just a *series*. We can add finitely many terms of the series to obtain

$$\begin{aligned} A_1 &= a_1, \\ A_2 &= a_1 + a_2, \\ A_3 &= a_1 + a_2 + a_3, \\ &\dots \\ A_n &= a_1 + a_2 + a_3 + \cdots + a_n, \\ &\dots \end{aligned}$$

The numbers $A_1, A_2, \dots, A_n, \dots$ are called the *partial sums* of the series (5.1). They naturally form a sequence $\{A_n\}$ of partial sums. If $A = \lim_{n\to\infty} A_n$ and A is a finite number, then the series $\sum a_n$ is called *convergent*, A is called its *sum*, and we write

$$A = \sum_{n=1}^{\infty} a_n.$$

If the sequence $\{A_n\}$ is divergent (i.e., A is infinite or does not exist), then the series (5.1) is also called *divergent*.

Example 5.1. Perhaps the simplest example of an infinite series is the so-called *geometric series*

$$a + aq + aq^2 + \cdots + aq^n + \cdots = \sum_{n=0}^{\infty} aq^{n-1}, \quad a \neq 0.$$

Its partial sum for $q \neq 1$ equals

$$A_n = \frac{a - aq^n}{1 - q}. \tag{5.2}$$

Indeed, by direct computation we obtain

$$A_n - qA_n = (a + aq + \cdots + aq^{n-1}) - q(a + aq + \cdots + aq^{n-1}) = a - aq^n,$$

from which equation (5.2) immediately follows.

By taking the limit in (5.2), we see that if $|q| < 1$ then the geometric series converges with the sum equal to $\frac{a}{1-q}$. If $|q| \geq 1$, then the series diverges. In particular, if $q = 1$, then $\lim A_n$ is either ∞ or $-\infty$, depending on the sign of a, and if $q = -1$, then the series takes the form

$$a - a + a - a + \ldots,$$

and the value of partial sums alternates between a and 0. To summarize, the geometric series converges if $|q| < 1$, and

$$a + aq + aq^2 + \cdots + aq^n + \cdots = \sum_{n=0}^{\infty} aq^{n-1} = \frac{a}{1-q}.$$

◇

Example 5.2. Consider the series

$$\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{1}{4} + \cdots .$$

This series resembles the geometric series, and we can try to find its sum using a similar technique. Let S_n be a partial sum of the first n terms. Then

$$S_n - \frac{1}{2}S_n = \left(\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n}\right) - \left(\frac{1}{2^2} + \frac{2}{2^3} + \frac{3}{2^4} + \cdots + \frac{n}{2^{n+1}}\right)$$

$$= \frac{1}{2} + \frac{2-1}{2^2} + \frac{3-2}{2^3} + \cdots + \frac{n-(n-1)}{2^n} - \frac{n}{2^{n+1}} = \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}\right) - \frac{n}{2^{n+1}}$$

The term in parentheses on the right-hand side of the above identity is the geometric series with $a = 1/2$ and $q = 1/2$, its partial sum was computed in the previous example:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = \frac{1/2 - 1/2(1/2)^n}{1 - 1/2} = 1 - (1/2)^n.$$

From this we conclude that

$$S_n = \frac{1 - (1/2)^n - n/2^{n+1}}{1/2} = 2 - \frac{1}{2^{n-1}} - \frac{n}{2^n} \tag{5.3}$$

It follows that S_n converges to 2 as $n \to \infty$. Thus,

$$\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n = 2.$$

◇

Example 5.3. Determine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

We estimate its partial sum:

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} > n \cdot \frac{1}{\sqrt{n}} = \sqrt{n}.$$

We see that the partial sums grow indefinitely as n goes to infinity. Thus this series diverges. ◇

Let $\displaystyle\sum_{n=1}^{\infty} a_n$ be a series, and $A_m = \displaystyle\sum_{n=1}^{m} a_n$ be the partial sum. The quantity

$$R_m = \sum_{n=1}^{\infty} a_n - A_m = \sum_{n=m+1}^{\infty} a_n \tag{5.4}$$

is called the *remainder* of the series. We first observe that the series $\sum a_n$ converges if and only if any remainder R_m converges (as a series). Therefore, we may remove any finite (possibly very large!) number of elements from the series without affecting its convergence (or divergence). Further, if the series $\sum a_n$ converges, then by taking limit in (5.4) as $m \to \infty$ we see that $R_m \to 0$. The next theorem gives a simple test to verify divergence of certain series.

Theorem 5.1. *If the series* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges, then* $\displaystyle\lim_{n\to\infty} a_n = 0$.

Proof. Let $A_n = a_1 + \cdots + a_n$. Then, since the series $\sum a_n$ converges, $\lim A_n$ exists as $n \to \infty$. Hence, $a_n = A_n - A_{n-1}$, and so $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(A_n - A_{n-1}) = 0$. $\square$

The *contrapositive* formulation of this theorem is sometimes called the *Test for Divergence*: if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum a_n$ diverges. For example, the series

$$\sum_{n=1}^{\infty} \frac{1}{1+s^n}$$

diverges for $0 < s \leq 1$ because $1/(1+s^n)$ does not converge to zero as $n \to \infty$. It is, however, wrong in general to conclude from the convergence of $\{a_n\}$ to zero that the series $\sum a_n$ converges. For instance, in Example 5.3 the series diverges, yet $\lim a_n = 0$.

Suppose now that the series $\sum a_n$ consists of positive terms. Then partial sums $\{A_n\}$ form an increasing sequence. If this sequence is bounded, then by the Monotone Convergence Theorem, it follows that the sequence of partial sums (and therefore the series) converges. On the other hand, if the sequence of partial sums is unbounded, then the series diverges. We illustrate this in the next example.

Consider the so-called *harmonic series*[1] given by

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}. \tag{5.5}$$

Indeed, starting from the third term we can divide the series into groups consisting of $2, 4, 8, \ldots, 2^k, \ldots$ terms in each group:

$$\underbrace{\frac{1}{3} + \frac{1}{4}}_{2} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{4} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{8} + \ldots.$$

[1]The reason for the name is that every term of the series is the *harmonic mean* of the two neighbouring terms. Recall that the harmonic mean of two numbers a and b equals $\frac{2ab}{a+b}$. The harmonic mean is an important notion in geometry and physics.

Each group adds up to a number bigger than 1/2. Therefore, if we denote by H_n the partial sum of the first n terms of the series, we see that

$$\begin{aligned}
&H_4 > 1/2 + 1/2 = 1\\
&H_8 > H_4 + 1/2 > 1 + 1/2 = 3/2\\
&H_{16} > H_8 + 1/2 > 3/2 + 1/2 = 2\\
&\dots\\
&H_{2^k} > k \cdot 1/2\\
&\dots
\end{aligned} \tag{5.6}$$

Thus the sequence of partial sums is unbounded, and the harmonic series diverges. We note that as n grows, the value of the partial sum of n terms grows rather slowly. For example, Euler calculated that $H_{1000} \approx 7.48$ and $H_{1000000} = 14.39$.

Let us consider a more general series of the form

$$1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{5.7}$$

where s is some positive real number. If $s = 1$, then (5.7) becomes the harmonic series. If $s < 1$, then the terms in (5.7) are bigger than the corresponding terms in (5.5), and so are the partial sums, hence, the series also diverges.

Now consider the case $s > 1$. We write $s = 1 + t$, where t is some positive number. We have

$$\frac{1}{(n+1)^s} + \frac{1}{(n+2)^s} + \cdots + \frac{1}{(2n)^s} < n \cdot \frac{1}{n^s} = \frac{1}{n^t}. \tag{5.8}$$

Splitting the series into groups, analogously to what we did for the harmonic series we have

$$\underbrace{\frac{1}{3^s} + \frac{1}{4^s}}_{2} + \underbrace{\frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \frac{1}{8^s}}_{4} + \underbrace{\frac{1}{9^s} + \cdots + \frac{1}{16^s}}_{8} + \dots.$$

From (5.8) it follows that each group above is less than the corresponding term of the geometric series

$$\left\{\frac{1}{2^t}, \frac{1}{4^t}, \frac{1}{8^t}, \dots\right\} = \left\{\frac{1}{2^t}, \frac{1}{(2^t)^2}, \frac{1}{(2^t)^3}, \ \dots\right\}$$

Since this geometric series $\{(\frac{1}{2^t})^n\}$ converges, we conclude that the sequence of partial sums of the series in (5.7) is bounded above, and therefore converges by the Monotone Convergence Theorem. Hence, the series (5.7) also converges. Another proof of convergence of the series for $s > 1$ will be given later, when we discuss the Integral Test for Convergence. The sums of this series are the values of a famous function $\zeta(s)$, called the Riemann ζ-function. It plays fundamental role in Number theory.

Example 5.4. Consider

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - n}.$$

First observe that $\frac{1}{n^2-n} = \frac{1}{n-1} - \frac{1}{n}$. Therefore, the partial sum A_n of this series equals

$$
\begin{aligned}
A_n &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\
&= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \cdots + \left(-\frac{1}{n-1} + \frac{1}{n-1}\right) - \frac{1}{n} = 1 - \frac{1}{n}.
\end{aligned} \tag{5.9}
$$

Thus $A_n \to 1$, and the series converges to 1. Series of this type are called *telescoping series.* $\diamond$

5.2. **Comparison Theorems.** Convergence or divergence of series can be often determined by comparing a given series to another series, which is known to converge or diverge. In the next theorems we assume that $\sum_n^\infty a_n$, and $\sum_n^\infty b_n$ are series with positive terms

Theorem 5.2 (Comparison Test)**.** *Suppose that there exists a number $N > 0$ such that the inequality $a_n \leq b_n$ holds for all $n > N$. Then convergence of $\sum b_n$ implies convergence of $\sum a_n$. Equivalently, divergence of $\sum a_n$ implies that of $\sum b_n$.*

Proof. We may drop any finite number of terms of the series without affecting its convergence. Therefore, without loss of generality we may assume that that $a_n \leq b_n$ for all $n = 1, 2, \ldots$. Denote by A_n, and B_n the partial sums of $\sum a_n$ and $\sum b_n$ respectively. Then $A_n \leq B_n$. Suppose that $\sum b_n$ converges. Then the sequence of partial sums $\{B_n\}$ is bounded above: $B_n \leq L$, for some $L > 0$. Therefore $A_n \leq B_n \leq L$, and by the Monotone Convergence Theorem, the sequence $\{A_n\}$ also converges. This proves the theorem. $\square$

Theorem 5.3 (Limit Comparison Test)**.** *Suppose there exists a limit*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = K \quad (0 \leq K \leq \infty).$$

Then:

(i) if the series $\sum b_n$ converges and $K < \infty$, then $\sum a_n$ converges.
(ii) if $\sum b_n$ diverges and $K > 0$, then $\sum a_n$ also diverges.

Proof. Suppose $\sum b_n$ converges with $K < \infty$. Given any $\epsilon > 0$ by definition of the limit, for sufficiently large n we have

$$\frac{a_n}{b_n} < K + \epsilon \implies a_n < (K + \epsilon) \cdot b_n$$

Since the series $\sum c_n = \sum (K + \epsilon) b_n$ obtained by multiplying the series $\sum b_n$ by a constant $(K + \epsilon)$ converges, we may apply the Comparison Test to $\sum a_n$ and $\sum c_n$ to conclude that the series $\sum a_n$ also converges.

The proof of the second statement is Exercise 6.5. $\square$

Theorem 5.4. *Suppose there exists $N > 0$ such that for $n > N$ we have*

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}. \tag{5.10}$$

Then convergence of $\sum b_n$ implies convergence of $\sum a_n$ and divergence of $\sum a_n$ implies that of $\sum b_n$.

The proof of this theorem is Exercise 6.7.

Example 5.5. Test for convergence the following series.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$,

(ii) $\sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$,

(iii) $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln\frac{n+1}{n}\right)$.

Solutions:

(i) $\frac{n!}{(2n)!} = \frac{1^2 2^2 3^2 \dots n^2}{1 \cdot 2 \cdot \dots \cdot n \cdot (n+1) \cdot \dots \cdot (2n)} = \frac{1}{n+1} \cdot \frac{2}{n+2} \cdot \dots \cdot \frac{n}{2n} < \frac{1}{2^n}$. Since $\sum \frac{1}{2^n}$ converges, it follows by the Comparison Test (Theorem 5.2) that the series $\sum \frac{n!}{(2n)!}$ also converges.

(ii) We use the Limit Comparison Theorem: Since

$$\frac{1}{n\sqrt[n]{n}} \div \frac{1}{n} = \frac{1}{\sqrt[n]{n}} \to 1,$$

and the harmonic series $\sum \frac{1}{n}$ diverges, we conclude that the series $\sum \frac{1}{n\sqrt[n]{n}}$ also diverges.

(iii) We use the inequality $\ln(1+x) \le x$, which holds for $-1 < x$ (See Lecture 1, Example. 1.12). First observer that

$$\ln\left(1 + \frac{1}{n}\right) < \frac{1}{n} \quad \Longrightarrow \quad 0 < \frac{1}{n} - \ln\left(\frac{n+1}{n}\right).$$

Furthermore,

$$-\ln\frac{n+1}{n} = \ln\frac{n}{n+1} = \ln\left(1 - \frac{1}{n+1}\right) < -\frac{1}{n+1}.$$

Therefore,

$$0 < \frac{1}{n} - \ln\frac{n+1}{n} < \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)} < \frac{1}{n^2}$$

Thus, the series converges by the Comparison Test and convergence of $\sum \frac{1}{n^2}$. $\diamond$

5.3. **Power series.** A power series *centred at a point* a is defined to be an expression of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots, \tag{5.11}$$

where c_j are some numbers. In particular, a power series centred at the origin (i.e., when $a = 0$) has the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots.$$

A power series centred at 0 can be thought of as an "infinite polynomial". The values of x for which the series (5.11) converges form *the domain of convergence* of the power series. It is clear from (5.11) that it always contains its centre a, and so on its domain of convergence the power series defines a function of x.

Example 5.6. Consider the power series centred at the origin given by

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots.$$

This is a geometric series with $a = 1$, $q = x$. The series converges for $|x| < 1$, and diverges for all other values of x. In fact, the function that this series defines on $|x| < 1$ is

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

We will show that the domain of convergence of a power series is always an interval (finite or infinite), or one point - its centre. For that we first prove the following

Lemma 5.5. *If the power series*

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \tag{5.12}$$

converges for some $x = b$, then it converges for all $|x| < |b|$.

Proof. Since the power series (5.12) converges for $x = b$, the sequence $\{c_n b^n\}$ is bounded, i.e., there exists $M > 0$ such that $|c_n b^n| < M$ for all n. Therefore, for any x satisfying $|x| < |b|$, we have

$$|c_n x^n| = \left|\frac{c_n x^n b^n}{b^n}\right| < M \left|\frac{x^n}{b^n}\right| = M \left|\frac{x}{b}\right|^n.$$

Since $|x/b| < 1$, the geometric series $\displaystyle\sum_{n=0}^{\infty} M \left|\frac{x}{b}\right|^n$ converges, and by the Comparison Test, so does the series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$. □

Example 5.7. Consider $\displaystyle\sum_{n=0}^{\infty} n! x^n$. By the Ratio Test

$$\lim_{n\to\infty} \frac{(n+1)!\ x^{n+1}}{n!\ x^n} = \lim_{n\to\infty} (n+1)\, x = \infty$$

for all values of x. Therefore, the series converges only for $x = 0$. $\diamond$

If the domain of convergence is unbounded, then it must equal all of $\mathbb{R}$. Indeed, for all points b in the domain of convergence, by Lemma 5.5 the series converges for all x which are less than $|b|$ in absolute value. By taking bigger and bigger b, we see that the series converges for all $x \in \mathbb{R}$.

On the other hand, suppose that the series in (5.12) has a point different from the origin where it converges, and that the domain of convergence is bounded. Consider the least upper bound of the set of values of x for which the series converges. Recall that the least upper bound exists by the Axiom of Completeness (see Lecture 4). Denote it by R. We claim that if $|x| < R$, then the series converges. Indeed, suppose on the contrary that the series diverges for this x. It follows from Lemma 5.5 that for $|x'| > |x|$ the series also diverges. So $|x|$ is an upper bound for the domain of convergence of the series. But this contradicts the fact that R is the least upper bound. This proves the claim. Similarly, if $|x| > R$, then the series diverges. Combining everything together yields the following

Theorem 5.6. *For the series*

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

one of the following mutually exclusive possibilities holds:

(i) The series converges at the origin only.

(ii) The series converges on the whole real line $\mathbb{R}$*.*

(iii) There exists $R > 0$ *such that the series converges for* $|x| < R$ *and diverges for* $|x| > R$*.*

The number R in (iii) is called *the radius of convergence* of the series. We will use the convention that $R = 0$ in (i), and $R = \infty$ in (ii). When possibility (iii) holds the power series may converge at the end points R and $-R$. Thus, the domain of convergence in this case is one of the following intervals: $(-R, R)$, $[-R, R)$, $(-R, R]$, or $[-R, R]$. Because of that, we will also call the domain of convergence *the interval of convergence* of the power series.

Corollary 5.7. *The power series*

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots.$$

either converges at one point a*, or on the whole of* $\mathbb{R}$*, or on one of the bounded interval* $(a-R, a+R)$*,* $[a-R, a+R)$*,* $(-aR, a+R]$*, or* $[a-R, a+R]$*, where* $R > 0$*.*

Example 5.8. Find the interval of convergence of the following power series:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-5)^n}{n\,2^n}$

Solution: (a) We apply the Ratio Test:

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} = \lim_{n\to\infty} \frac{x}{n+1} = 0$$

for all x. Thus, the interval of convergence of this power series is $\mathbb{R}$.

(b) Again by the Ratio test,

$$\lim_{n\to\infty} \frac{|x-5|^{n+1}}{(n+1)\,2^{n+1}} \cdot \frac{n\,2^n}{|x-5|^n} = \frac{|x-5|}{2}.$$

For the convergence of the series we need to have $\frac{|x-5|}{2} < 1$, or

$$|x-5| < 2.$$

Thus the radius of convergence equals 2. To determine the interval of convergence we need to investigate the end points. When $x = -3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(3-5)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n},$$

which converges by the Alternating Series Test. For the other end point $x = 7$ we have

$$\sum_{n=1}^{\infty} \frac{(7-5)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{1}{n},$$

which a harmonic series, and so diverges. Thus the interval of convergence of the series is $[3, 7)$. $\diamond$

Sometimes it is convenient to find the radius of convergence R of the power series $\sum c_n(x-a)^n$ using the so-called *Cauchy-Hadamard formula*:

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|c_n|}, \tag{5.13}$$

provided that the limit exists. (If the limit does not exist, then one can show that $1/R$ equals the *largest* limit of $\sqrt[n]{|c_n|}$, that is the limit as n approaches infinity of the supremum of the elements of the sequence after the n-th position.)

Example 5.9. To find the interval of convergence of the series

$$\sum_{n=1}^{\infty} 3^n x^n$$

we use formula (5.13):

$$\lim_{n\to\infty} \sqrt[n]{|3^n|} = 3,$$

and so $R = 1/3$. A simple calculation shows that the series diverges at both end points, $x = 1/3$ and $x = -1/3$. Thus the interval of convergence is $(-1/3, 1/3)$. $\diamond$

One can show that on the (open) interval of convergence, the power series (5.11) is continuous and differentiable. The proof of these facts, however, goes beyond our capabilities at this point. Furthermore, the convergent power series can be differentiated and integrated term-by-term. This will produce the functions which are respectively the derivative and the anti-derivative of the function defined by the original series. More precisely, the following holds.

Theorem 5.8. *Suppose the power series $\sum_n c_n(x-a)^n$ has the radius of convergence $R > 0$, and on the interval $(a-R, a+R)$ it defines the function $f(x)$:*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n.$$

Then

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}, \tag{5.14}$$

and

$$\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}. \tag{5.15}$$

Furthermore, the radius of convergence is the same for all three power series above.

The theorem can be proved by applying the Root Test to the series in (5.14) and (5.15), and using the fact that $\sqrt[n]{n} \to 1$ as $n \to \infty$. It should be noted that in the theorem above, while the radius of convergence is the same, the *interval of convergence* can be different for $f(x), f'(x)$ and $\int f(x)dx$.

Example 5.10. Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} nx^n,$$

and compute the function that it represents.

Solution: we consider the series $\sum_{n=1}^{\infty} x^n$. For every fixed x with $|x| < 1$, this is a geometric series converging to $\frac{x}{1-x}$:

$$\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots = \frac{x}{1-x}$$

Differentiating this series term-by-term gives

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Therefore, we conclude that

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

By Theorem 5.8 this series has the radius of convergence 1. By inspection, the series diverges at both end points $x = 1$ and $x = -1$. Thus the interval of convergence of the series is $(-1, 1)$. ⋄

Example 5.11. Find the power series representation for the functions $f(x) = \frac{1}{2+x}$ centred at (a) the origin, (b) $a = 1$.

(a) We have

$$\frac{1}{2+x} = \frac{1}{2} \cdot \frac{1}{1-(-x/2)} = \frac{1}{2} \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^{n+1}} = \frac{1}{2} - \frac{x}{4} + \frac{x^2}{8} - \frac{x^3}{16} + \ldots.$$

Using the ratio test one can see that the interval of convergence of this power series is $(-2, 2)$.

(b) The power series representation centred at $a = 1$ will have the powers of $(x - 1)$. Therefore, we should try to use the geometric series that will have the powers of $(x - 1)$. We have

$$\frac{1}{2+x} = \frac{1}{3+(x-1)} = \frac{1}{3} \cdot \frac{1}{1-(-\frac{x-1}{3})} = \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{x-1}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{3^{n+1}}.$$

Again, using the Ratio Test, we see that the radius of convergence is 3, and the interval of convergence is $(-2, 4)$ Note that although the power series in (a) and (b) are different, and have different intervals of convergence, the point $x = -2$ is the end point for both of them. This is because the function $f(x)$ is not defined at this point. ⋄

Example 5.12. To find the power series representation of the function $\ln(1 + x^2)$ centred at the origin, first consider the function $\ln(1 + x)$. Its derivative equals $\frac{1}{1+x}$ with

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^n.$$

Integrating the above expression term-by-term we obtain

$$\ln(1+x) = \int \frac{dx}{1+x} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + C = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} + C.$$

To determine the value of C, we set $x = 0$ which yields $0 = \ln(1+0) = C$. Therefore,

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}. \tag{5.16}$$

Finally, by replacing x with x^2 in the above formula, we obtain:

$$\ln(1+x^2) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x^2)^n}{n} = x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \dots.$$

$\diamond$

Example 5.13. Prove that

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots.$$

Solution: Substitute $x = 1$ into formula (5.16) to get the result.

Note: This seemingly simple solution, however, requires a justification: why is the power series continuous (from one side) at the end point of the interval of convergence, provided that the series converges at that point? This is the content of the so called Abel's theorem, the proof of which can be found in advanced calculus textbooks. $\diamond$

5.4. **Taylor Series.** Let $f(x)$ be a function that has derivatives of all orders on the interval $(a - R, a + R)$ for some $a \in \mathbb{R}$, and $R > 0$. Suppose that $f(x)$ can be represented on $(a - R, a + R)$ by a convergent power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n. \tag{5.17}$$

This means that for any $x \in (a - R, a + R)$, the series (5.17) converges to $f(x)$. Then by direct differentiation of the power series (5.17), we see that $f^{(n)}(a) = n!\, c_n$, for all $n > 0$ (here $f^{(n)}$ denotes the derivative of $f(x)$ of order n). From this we conclude that

$$c_n = \frac{f^{(n)}(a)}{n!},$$

and thus the series in (5.17) becomes

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \tag{5.18}$$

This is called the *Taylor series centred at $x = a$ associated with* $f(x)$. If $a = 0$, then (5.18) becomes

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots, \tag{5.19}$$

which is called the *Maclaurin series* associated with $f(x)$.

Example 5.14. Let $P(x)$ be a polynomial of degree N,

$$P(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_N x^N.$$

By inspection, $c_n = \dfrac{P^{(n)}(0)}{n!}$ for $n = 1, \dots N$, and $c_n = 0$ for $n > N$. Thus, the Maclaurin series associated with $P(x)$ is exactly $P(x)$. $\diamond$

In general, however, one cannot immediately conclude that the Taylor or Maclaurin series associated with a function $f(x)$ converges to $f(x)$. In fact, it is not even clear whether the Taylor series of a given function converges at all. (Note that when we derived (5.18) we assumed to begin with that $f(x)$ has a power series representation.) Define the *Taylor polynomial* to be

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(N)}(a)}{N!}(x-a)^N, \tag{5.20}$$

i.e., $T(x)$ is simply the order N partial sum of the Taylor series (5.18). Thus, by the definition of convergence, in order to show convergence of the Taylor series to $f(x)$ we need to show that

$$\lim_{N\to\infty} T_N(x) = f(x) \tag{5.21}$$

for all x on some interval. If we define the *remainder* of the Taylor series to be

$$R_N(x) = f(x) - T_N(x) \tag{5.22}$$

then proving (5.21) is equivalent to showing

$$R_N(x) \to 0, \quad \text{as} \quad N \to \infty.$$

The following theorem provides a useful tool for proving convergence of Taylor series. For simplicity, we consider the case when $a = 0$. Then $T_N(x) = f(0) + f'(0)x + \dots \frac{f^{(N)}(0)}{N!}x^N$, and

$$R_N(x) = f(x) - \left(f(0) + f'(0)x + \dots \frac{f^{(N)}(0)}{N!}x^N \right) \tag{5.23}$$

Theorem 5.9 (Lagrange's Remainder Theorem)**.** *Let f be infinitely differentiable on $(-R, R)$. Then there exists a number c satisfying $|c| < |x|$ such that*

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}. \tag{5.24}$$

Example 5.15. Let $f(x) = e^x$. Then $f^{(n)}(0) = e^0 = 1$ for all n. Therefore, $c_n = \frac{1}{n!}$, and we have

$$e^x \sim \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

The remainder of order N of this Maclaurin series is

$$R_N = e^x - \left(\frac{x^{N+1}}{(N+1)!} + \frac{x^{N+2}}{(N+2)!} + \dots \right).$$

According to Lagrange's Remainder Theorem, there is a number c, $|c| < |x|$, such that

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1} = \frac{e^c}{(N+1)!}x^{N+1}.$$

For any fixed x, $R_N(x) \to 0$, since for any x, $\frac{x^n}{n!} \to 0$ as $n \to \infty$. Thus

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots.$$

for all $x \in \mathbb{R}$. $\diamond$

Example 5.16. Let

$$g(x) = \begin{cases} e^{-1/x^2}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases} \tag{5.25}$$

Since e^{-1/x^2} approaches 0 as $x \to 0$, the function $g(x)$ is continuous at 0. In fact, one can show that $g(x)$ has continuous derivatives of any order at $x = 0$, and $g^{(n)}(0) = 0$ for any $n > 0$. Indeed, to show that $g'(0) = 0$ first observe that

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0^+} \frac{e^{-1/h^2}}{h}.$$

(Note that $g(h) = 0$ for $h < 0$, so we may assume that $h > 0$.) Consider first

$$\lim_{h \to 0^+} \frac{e^{-1/h}}{h} = \lim_{h \to 0^+} \frac{1/h}{e^{1/h}} = \lim_{h \to 0^+} \frac{-1/h^2}{e^{1/h}(-1/h^2)} = \lim_{h \to 0^+} \frac{1}{e^{1/h}} = 0.$$

Here we used L'Hôpital's Rule. Next, observer that

$$0 < e^{-1/h^2} < e^{-1/h},$$

and thus by the Squeeze theorem we have $\lim_{h \to 0^+} \frac{e^{-1/h^2}}{h} = 0$, which proves that $g'(0) = 0$. Analogous proof will work for arbitrary n.

The Maclaurin series associated to $g(x)$ is, therefore, identically zero. It follows that the Maclaurin series associated with $g(x)$ does not converge to $g(x)$ for $x > 0$. $\diamond$

Definition 5.10. *An infinitely differentiable function $f(x)$ is called real-analytic in a neighbourhood of a point $x = a$, if for some positive R the Taylor series* (5.18) *associated with $f(x)$ converges to $f(x)$ on $(a - R, a + R)$.*

Thus, e^x is a real-analytic function, while the function $g(x)$ in Example 5.16 is not real analytic near $x = 0$.

Proof of Lagrange's Remainder Theorem. . First note the following version of the Mean Value Theorem: If $g(x)$ and $h(x)$ are continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b) and $h'(x) \neq 0$, then there exists a point $c \in (a, b)$ such that

$$\frac{g(b) - g(a)}{h(b) - h(a)} = \frac{g'(c)}{h'(c)} \tag{5.26}$$

This can be proved by applying the Mean Value Theorem to the function

$$\phi(x) = (g(b) - g(a))h(x) - (h(b) - h(a))g(x).$$

Note that the n-th order derivative of $R_N(x)$ at $x = 0$ vanishes for $n = 0, 1, 2, \ldots, N$ (see Exercise 6.9). Therefore, if we apply (5.26) to functions $g(x) = R_N(x)$ and $h(x) = x^{N+1}$, then (assume $x > 0$ for simplicity) there exists a point $c_1 \in (0, x)$ such that

$$\frac{R_N(x)}{x^{N+1}} = \frac{R'_N(c_1)}{(N+1)c_1^N}.$$

We now repeat the process and apply (5.26) to functions $g(x) = R'_N(x)$ and $h(x) = x^N$ on the interval $(0, c_1)$: there is $c_2 \in (0, c_1)$ such that

$$\frac{R'_N(c_1)}{c_1^N} = \frac{R''_N(c_2)}{Nc_2^{N-1}}.$$

Continue the process inductively N times. In the end we get

$$R_N(x) = \frac{x^{N+1}}{(N+1)!} \frac{R_N^{(N+1)}(c_{N+1})}{c_{N+1}^{N-N}},$$

where $c_{N+1} \in (0, c_N) \subset \cdots \subset (0, x)$. Now set $c = c_{N+1}$, then $c^{N-N} = 1$, and we can write

$$R_N(x) = \frac{R_N^{(N+1)}(c)}{(N+1)!} x^{N+1} = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1},$$

where the last equality follows from the fact that $R_N^{(N+1)}(x) = (f(x) - T_N(x))^{(N+1)} = f^{(N+1)}(x)$, because $T_N^{(N+1)} \equiv 0$. This proves the theorem. □

Example 5.17. Let $f(x) = (1+x)^{1/2}$. Then

$$f^{(n)}(0) = \frac{1}{2}\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - 2\right) \ldots \left(\frac{1}{2} - n + 1\right).$$

Therefore,

$$c_n = \binom{1/2}{n} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots\left(\frac{1}{2}-n+1\right)}{n!},$$

and hence

$$(1+x)^{1/2} \sim \sum_{n=0}^{\infty} \binom{1/2}{n} x^n$$

is the associated Maclaurin series. This is called the *binomial series.* Let us try use Lagrange's Remainder Theorem again to determine convergence of the series above. We have

$$R_N(x) = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots\left(\frac{1}{2}-N\right)(1+c)^{1/2-N}}{(N+1)!} x^{N+1}$$

for some c, $|c| < |x|$. If $|x| < 1$, then clearly $x^{N+1} \to 0$ as $N \to \infty$. Also, $\lim_{N\to\infty} \binom{1/2}{N} = 0$ (see Exercise 6.13). If $c > 0$, then we also have $(1+c)^{1/2-N} \to 0$ as $N \to \infty$. However, if $c < 0$, then $(1+c)^{1/2-N}$ does not go to zero, and we cannot be sure that $R_N(x)$ goes to zero.

In general, the binomial series converges for $x \in (-1, 1)$, and we have

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n, \quad k \in \mathbb{R}, \quad \text{and } |x| < 1,$$

where the *binomial coefficients* are defined by

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}.$$

◇

Exercises

5·1 Use the technique of Example 5.2 to find the values of q for which the series

$$\sum_{n=1}^{\infty} nq^n$$

converges.

5·2 Prove that if the series $\sum a_n$ converges then its remainder R_m as defined in (5.4) converges to zero.

5·3 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+s^n}$ converges or diverges for $s > 1$.

5·4 Find the sum of the series if it is converging: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$.

5·5 Prove part (ii) of Theorem 5.3.

5·6 Test for convergence the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+2)}}$,

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$,

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^p}$, $p > 0$,

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\ln n)^{\ln n}}$.

5·7 Prove Theorem 5.4. Hint: multiply equations (5.10) term by term, and use Theorem (5.2).

5·8 Formulate Lemma 5.5 for the power series given by equation (5.11).

5·9 Prove that in the proof of Lagrange's theorem the remainder $R_N(x)$ given by equation (5.23) satisfies $R_N^{(n)}(0) = 0$ for all $n = 0, 1, \ldots, N$.

5·10 Formulate Theorem 5.6 for the power series given by equation (5.11).

5·11 Show that the function $g(x)$ in Example 5.16 satisfies $g'(0) = 0$.

5·12 Use Lagrange's Remainder Theorem to prove that the Maclaurin series of $\cos x$ converges to $\cos x$ for all x.

5·13 Show that for any m,

$$\lim_{n\to\infty} \binom{m}{n} = 0.$$