First Order Differential Equations

Definition 10.1. A *differential equation* is an equation which relates some unknown function, say $y$, to one or more of its derivatives.

Definition 10.2. A *first order* differential equation involves only the first derivative of the function. A higher order differential equation involves higher derivatives. For instance, a differential equation relating $y$ to its second derivative (and perhaps its first derivative as well) would be a *second order* differential equation.

Examples:

$$y \frac{dy}{dx} = x$$ and $$y - 3 \frac{dy}{dx} + 4 = 0$$ are first order differential equations; also, $$y + \frac{dy}{dx} - \frac{d^2y}{dx^2} = 3x^2 - 2$$ is a second order differential equation.

We will sometimes use the abbreviation D.E. for the phrase differential equation.

Definition 10.3. A *solution* to a D.E. is a function of the form $y = f(x)$ (i.e., an expression for $y$ which does not involve $y$ or its derivatives), which satisfies the differential equation. We often denote such a solution as $y(x)$.

Generally, there are many solutions to a differential equation. For instance, the second order D.E. $y = -\frac{d^2y}{dx^2}$ is satisfied by $y = \sin x$ or by $y = \cos x$, or in fact by $y = A \sin x + B \cos x$ for any values of $A$ and $B$. To “solve” a D.E.
usually means to find the General Solution, that is, to find the form that all solutions to the D.E. have.

We will only be concerned with first order differential equations. We will study two types of first order differential equations, namely *separable*, and *linear* differential equations. We need different approaches for solving these 2 different types of D.E.’s.

**Definition 10.4.** A *separable* first order differential equation is an equation which may be put into one of the following forms:

\[ \frac{dy}{dx} = f(x) \quad \text{or} \quad \frac{dy}{dx} = \frac{g(y)}{f(x)} \quad \text{or} \quad \frac{dy}{dx} = f(x)g(y) \]

Strictly speaking, we have already dealt with some separable D.E.’s, as the first example will show. First, though, we outline some basic steps for solving such a problem.

**Steps For Solving a Separable Differential Equation:**

(Assuming that the D.E. is given in terms of \( \frac{dy}{dx} \))

1. Recognize the problem as a separable D.E.
   e.g. we have \( \frac{dy}{dx} = \frac{f(x)}{g(y)} \)

2. Separate the equation into the form
   (terms involving \( y \))(\( dy \)) = (terms not involving \( y \))(\( dx \))
   e.g. rearrange \( \frac{dy}{dx} = \frac{f(x)}{g(y)} \) to \( g(y)dy = f(x)dx \)

3. Integrate both sides, adding one arbitrary constant, say C, to the \( x \) side. (This is done because adding an arbitrary constant to both sides after integrating is equivalent to adding just a single arbitrary constant to one side.)
   e.g. evaluate \( \int g(y)dy = \int f(x)dx \)

4. Solve for \( y \) if possible.
   If we can solve for \( y \) we get what is called the *general solution*, and if we cannot solve for \( y \) we get what is called an *implicit (general) solution*. 
Example 1. Find the general solution to \( \frac{dy}{dx} = x^2 \).

**Solution:** First we recognize that we have a separable differential equation. That is, we have \( \frac{dy}{dx} = \frac{f(x)}{g(y)} \) where \( f(x) = x^2 \) and \( g(y) = 1 \). We rearrange (i.e. separate) the given equation to:

\[
dy = x^2 \, dx
\]

and then integrate both sides:

\[
\int dy = \int x^2 \, dx \quad \Rightarrow \quad y = \frac{x^3}{3} + C
\]

We have found the general solution to \( \frac{dy}{dx} = x^2 \) as:

\[
y = \frac{x^3}{3} + C
\]

That is, any function of this form is a solution to the differential equation.

Example 2. Solve \( \frac{dy}{dx} = \frac{4x}{3y^3} \).

**Solution:** We first recognize this as a separable D.E., with \( f(x) = 4x \) and \( g(y) = 3y^3 \). Proceeding with our steps, we get:

\[
3y^3 \, dy = 4x \, dx \quad \text{step (2): separate}
\]

\[
\Rightarrow \int 3y^3 \, dy = \int 4x \, dx \quad \text{Begin step(3): set up integration}
\]

\[
\Rightarrow 3 \left( \frac{y^4}{4} \right) = 4 \left( \frac{x^2}{2} \right) + C \quad \text{step(3): integrate}
\]

\[
\Rightarrow \frac{3y^4}{4} = 2x^2 + C \quad \text{Begin step(4): isolate } y \text{ if possible}
\]

\[
\Rightarrow y^4 = \frac{4}{3} [2x^2 + C]
\]

\[
\Rightarrow y^4 = \frac{8}{3} x^2 + \frac{4}{3} C
\]

\[
\Rightarrow y = \pm \sqrt[4]{\frac{8}{3} x^2 + C_1} \quad (\frac{4}{3} C \text{ is still an arbitrary constant, which we can call } C_1)
\]

We were able to find a general solution, namely \( y = \pm \sqrt[4]{\frac{8}{3} x^2 + C_1} \).
Example 3. Solve \( y \frac{dy}{dx} = 4x^3 + 2x \).

Solution: We recognize this as a separable D.E., with \( f(x) = 4x^3 + 2x \) and \( g(y) = e^y \). We finish separating the equation and proceed:

\[
e^y \, dy = (4x^3 + 2x) \, dx \quad \text{step (2)}
\]

\[
\Rightarrow \int e^y \, dy = \int (4x^3 + 2x) \, dx \quad \text{Begin step (3)}
\]

\[
\Rightarrow e^y = x^4 + x^2 + C \quad \text{step (3)}
\]

\[
\Rightarrow \ln(e^y) = \ln(x^4 + x^2 + C) \quad \text{Begin step (4)}
\]

\[
\Rightarrow y = \ln(x^4 + x^2 + C) \quad \text{step (4)}
\]

Once again we have a general solution, namely \( y = \ln(x^4 + x^2 + C) \).

Example 4. Solve \( \frac{dy}{dx} = \frac{\cos x}{y + e^y} \).

We see that we have a separable D.E., so we proceed as usual:

\[
(y + e^y) \, dy = \cos x \, dx
\]

\[
\Rightarrow \int (y + e^y) \, dy = \int \cos x \, dx
\]

\[
\Rightarrow \frac{y^2}{2} + e^y = \sin x + C
\]

Here, it would be very difficult, or even impossible, to isolate \( y \), so we cannot put this into \( y = y(x) \) form to get our usual \textit{explicit} general form of the solution. However, the equation we have found, relating \( x \) and \( y \), is an \textit{implicit} general solution to the given D.E., that is, implicitly defines the family of functions \( y = y(x) \) which satisfy the differential equation.

Differential Equations with Side Conditions

Recall that we used side conditions with integration to allow the constant of integration, \( C \), to be found. Similarly we may use such conditions to solve
for C here. That is, we may have some further information about y which allows us to find the specific one among the family of solutions (represented by the general solution) that we’re interested in.

**Example 5.** Solve $\frac{dy}{dx} = \frac{y}{x^2}$, subject to the condition that when $x = \frac{1}{2}$, $y = e$.

**Solution:** We solve the D.E. as usual, and then use the side condition that $y = e$ when $x = \frac{1}{2}$, to solve for the constant.

Clearly, since $y = e$ when $x = \frac{1}{2}$, then the function we are looking for is not the function $y = 0$, so we can divide through by $y$ to get:

$$\int \frac{1}{y} dy = \int \frac{1}{x^2} dx$$

$$\Rightarrow \ln |y| = -\frac{1}{x} + C$$

$$\Rightarrow \ln |y| = -\frac{1}{x} + C$$

$$\Rightarrow e^{\ln |y|} = e^{-\frac{1}{x} + C}$$

$$\Rightarrow |y| = e^{-\frac{1}{x}} e^C$$

$$\Rightarrow y = \pm e^C e^{-\frac{1}{x}}$$

$$\Rightarrow y = Ae^{-\frac{1}{x}}$$

In this general solution, the arbitrary constant $A = \pm e^C$ can have any value except 0. We now use the fact that when $x = \frac{1}{2}$, $y = e$ to find the specific function we require. We have:

$$e = Ae^{-2} \Rightarrow e^3 = A$$

and substituting $A = e^3$ into the general solution, we get:

$$y = e^3 e^{-\frac{1}{x}} = e^{(3-\frac{1}{x})}$$

So here we have the *specific* solution, $y = e^{(3-\frac{1}{x})}$
Example 6. Find the solution to \( \frac{dy}{dx} = e^x - y \) for which \( y(0) = \ln 2 \).

First, we need to rewrite the differential equation in a form in which we can recognize it as separable. We know that \( e^x - y = e^x e^{-y} \), so we have:

\[
\frac{dy}{dx} = \frac{e^x}{e^y}
\]

We separate and integrate:

\[
e^y dy = e^x dx \Rightarrow \int e^y dy = \int e^x dx \Rightarrow e^y = e^x + C
\]

To isolate \( y \), we take natural logarithms of both sides:

\[
\ln (e^y) = \ln (e^x + C) \Rightarrow y = \ln (e^x + C)
\]

Now, we use our side condition \( y(0) = \ln 2 \). We see that:

\[
y(0) = \ln (e^x + C)|_{x=0} = \ln (e^0 + C) = \ln (1 + C)
\]

so we must have \( \ln(1 + C) = \ln 2 \), i.e. \( 1 + C = 2 \Rightarrow C = 1 \).

Thus, the specific solution is \( y = \ln (e^x + 1) \).

That is, we see that \( y = \ln (e^x + 1) \) is the one function among the family of solutions to \( \frac{dy}{dx} = e^x - y \) which has \( y(0) = \ln 2 \).

We now move on to the second type of D.E. mentioned above, linear differential equations.

Definition 10.5. A linear differential equation is a D.E. which can be put into the form:

\[
\frac{dy}{dx} + p(x)y = q(x)
\]

where \( p(x) \) and \( q(x) \) are functions of only \( x \).

The approach we use for solving linear D.E.’s is very different from the one we used for separable D.E.’s. In order to state our solution method, we need the following definition.
Definition 10.6. Given a linear D.E.

\[ \frac{dy}{dx} + p(x) y = q(x) \]

we define the *integrating factor* \( \Phi(x) \) for the differential equation to be:

\[ \Phi(x) = e^{\int p(x) \, dx} \]

*Note:* When we calculate \( \int p(x) \, dx \) here, we let the integration constant \( C \) be zero.

The use of the integrating factor is detailed in the following, but often these steps are skipped (see next theorem).

We begin with a linear D.E.:

\[ \frac{dy}{dx} + p(x) y = q(x) \]

We multiply through the equation by the integrating factor, \( \Phi(x) = e^{\int p(x) \, dx} \), to get:

\[ e^{\int p(x) \, dx} \frac{dy}{dx} + e^{\int p(x) \, dx} p(x) y = q(x) e^{\int p(x) \, dx} \]

*Notice:* The left hand side of this equation is the derivative of \( ye^{\int p(x) \, dx} \). That is, if we differentiate \( ye^{\int p(x) \, dx} \) with respect to \( x \), we use the product rule and implicit differentiation to get:

\[
\frac{d}{dx} \left[ ye^{\int p(x) \, dx} \right] = \left( \frac{dy}{dx} \right)e^{\int p(x) \, dx} + \left[ e^{\int p(x) \, dx} \left( \frac{d}{dx} \int p(x) dx \right) \right] y
\]

and of course \( \frac{d}{dx} \int p(x) \, dx = p(x) \).

So after we multiply through the D. E. by the integrating factor, what we have is:

\[ \frac{d}{dx} \left[ ye^{\int p(x) \, dx} \right] = q(x) e^{\int p(x) \, dx} \]

We now integrate both sides. *Notice:* This time, we are integrating with respect to \( x \) on both sides of the equation.

\[ \int \left[ \frac{d}{dx} \left( ye^{\int p(x) \, dx} \right) \right] \, dx = \int \left( q(x) e^{\int p(x) \, dx} \right) \, dx \]
Of course, for any function \( f(x) \), \( \int \frac{d}{dx} [f(x)] \ dx = f(x) + C \) (but we omit the constant on the left hand side as before). Recognizing this, we see that we have:

\[
y e^{\int p(x) \ dx} = \int q(x) e^{\int p(x) \ dx} \ dx
\]

Now, we divide through by \( e^{\int p(x) \ dx} = \Phi(x) \), to isolate \( y \):

\[
y = e^{-\int p(x) \ dx} \int q(x) e^{\int p(x) \ dx} \ dx
\]

\[\Rightarrow y = \frac{1}{\Phi(x)} \int q(x) \Phi(x) \ dx\]

**Notice:** Although we can move constant multipliers inside or outside and integral, we **cannot** move a function of \( x \) across an integral symbol (where the integration is with respect to \( x \)). The \( \frac{1}{\Phi(x)} \) multiplier outside the integral cannot be used to cancel the \( \Phi(x) \) multiplier inside the integral.

The last equation above is the general solution to the differential equation. We now restate the above result as a theorem. We can use this result to solve a linear D.E. without having to go through the steps each time.

**Theorem 10.7.** Given a linear differential equation

\[
\frac{dy}{dx} + p(x) y = q(x)
\]

with integrating factor \( \Phi(x) = e^{\int p(x) \ dx} \), the general solution is given by:

\[
y = \frac{1}{\Phi(x)} \int q(x) \Phi(x) \ dx
\]

**Example 7.** Solve \( \frac{dy}{dx} + \frac{2y}{x} = 5x^2 \).

**Solution:** Notice that this is a linear first order differential equation, with \( p(x) = \frac{2}{x} \) and \( q(x) = 5x^2 \). By the previous theorem we need only find \( \Phi(x) \), and then apply the formula.

We get \( \int p(x) \ dx = \int \frac{2}{x} \ dx = 2 \int \frac{1}{x} \ dx = 2 \ln |x| + C = \ln x^2 + C = \ln x^2 \) (since we use \( C = 0 \) here). So the integrating factor is given by

\[
\Phi(x) = e^{\int p(x) \ dx} = e^{\ln(x^2)} = x^2
\]
Substituting into the general solution formula we can find the general solution:

\[ y = \frac{1}{\Phi(x)} \int q(x)\Phi(x)dx \]

\[ = \frac{1}{x^2} \int (5x^2)(x^3)dx \]

\[ = \frac{1}{x^2} \int 5x^4dx \]

\[ = \frac{1}{x^2} (x^5 + C) \quad \text{REMEMBER YOUR BRACKETS!} \]

So we see that the general solution is: 

\[ y = \frac{x^5}{x^2} + \frac{C}{x^2} = x^3 + \frac{C}{x^2} \]

*Notice:* \( \frac{1}{x^2} \) is **not** a constant and so cannot be absorbed into the arbitrary constant.

**Example 8.** Solve: \( \frac{dy}{dx} + y = e^x \)

**Solution:** We need only find \( p(x), q(x), \) and \( \Phi(x), \) and then apply the formula. We see that \( p(x) = 1 \) and \( q(x) = e^x, \) so we have \( \int p(x)dx = \int (1)dx = x + C \) and letting \( C = 0, \) the integrating factor is given by \( \Phi(x) = e^{\int p(x)dx} = e^x. \)

Substituting into the general solution formula we then have:

\[ y = \frac{1}{\Phi(x)} \int q(x)\Phi(x)dx \]

\[ = \frac{1}{e^x} \int (e^x)(e^x)dx \]

\[ = e^{-x} \int e^{2x}dx \quad \text{Note: the } e^{-x} \text{ multiplier cannot be} \]

\[ = e^{-x}(\frac{e^{2x}}{2} + C) \]

\[ = \frac{e^{2x} - x}{2} + Ce^{-x} \]

\[ = \frac{e^x}{2} + \frac{C}{e^x} \]

We see that the general solution to \( \frac{dy}{dx} + y = e^x \) is \( y = \frac{e^x}{2} + \frac{C}{e^x}. \)
Example 9. Solve $\frac{dy}{dx} - \frac{y}{x} = x$.

Here, we have $p(x) = -\frac{1}{x}$ and $q(x) = x$. We have

$$\int p(x)dx = \int \left(-\frac{1}{x}\right)dx = -\int \frac{1}{x}dx = -\ln|x|+C = \ln(|x|)^{-1}+C = \ln\left(\frac{1}{|x|}\right)+C$$

but we use $C = 0$, so the integrating factor is

$$\Phi(x) = e^{\ln\left(\frac{1}{|x|}\right)} = \frac{1}{|x|}$$

Thus we get:

$$y = \frac{1}{\Phi(x)} \int q(x)\Phi(x)dx = \left(\frac{1}{|x|}\right)^{-1} \int x \left(\frac{1}{|x|}\right)dx = |x| \int \frac{x}{|x|}dx$$

Notice: We can get rid of the absolute value signs here because either $|x| = x$, or else $|x| = -x$, in which case we are multiplying the integral by the constant $-1$ twice (once outside the integral and once inside, which can be brought out), i.e. we are multiplying by the constant $(-1)^2 = 1$. So we have

$$y = x \int \frac{x}{x}dx = x \int 1dx = x(x+C) = x^2 + Cx$$

Thus, the general solution to $\frac{dy}{dx} - \frac{y}{x} = x$ is $y = x^2 + Cx$.

Of course, we can have side conditions with linear D.E.’s, just as we did with separable D.E.’s.

Example 10. Solve $\frac{dy}{dx} = (\sin x)(1-y)$, subject to the condition $y\left(\frac{\pi}{2}\right) = 4$.

We can solve this as a linear D.E. if we first rearrange it:

$$\frac{dy}{dx} = (\sin x)(1-y) = \sin x - y\sin x$$

$$\Rightarrow \frac{dy}{dx} + (\sin x)y = \sin x$$

Now, we see that we have $p(x) = \sin x$ and also $q(x) = \sin x$. For the integrating factor, we have

$$\Phi(x) = e^{\int p(x)dx} = e^{\int \sin x dx} = e^{-\cos x}$$
Thus we get:

\[
y = \frac{1}{\Phi(x)} \int q(x)\Phi(x)dx
\]

\[
= \frac{1}{e^{-\cos x}} \int (\sin x) e^{-\cos x}dx
\]

\[
= e^{\cos x} \int e^u du \quad (\text{where } u = -\cos x \text{ so } du = \sin xdx)
\]

\[
= e^{\cos x} (e^{-\cos x} + C)
\]

\[
= e^{\cos x - \cos x} + Ce^{\cos x}
\]

\[
= 1 + Ce^{\cos x}
\]

We have the general solution \(y = 1 + Ce^{\cos x}\) and we know that \(y(\frac{\pi}{2}) = 4\), so we have

\[
y|_{x=\frac{\pi}{2}} = 4 \quad \Rightarrow \quad 1 + Ce^{\cos\left(\frac{\pi}{2}\right)} = 4
\]

\[
\Rightarrow \quad 1 + Ce^0 = 4 \quad (\text{since } \cos\left(\frac{\pi}{2}\right) = 0)
\]

\[
\Rightarrow \quad 1 + C = 4
\]

\[
\Rightarrow \quad C = 3
\]

We see that the specific solution is \(y = 1 + 3e^{\cos x}\).

**Example 11.** Solve \(\frac{dy}{dx} + 2xy = x^3\), subject to \(y(1) = 0\).

**Solution:** We have \(p(x) = 2x\) and \(q(x) = x^3\). Thus the integrating factor is:

\[
\Phi(x) = e^{\int p(x)dx} = e^{\int 2xdx} = e^{x^2}
\]

Then our formula for the general solution for a linear D. E. gives:

\[
y = \frac{1}{\Phi(x)} \int q(x)\Phi(x)dx = \frac{1}{e^{x^2}} \int x^3e^{(x^2)}dx
\]
To find $\int x^3e(x^2)\,dx$ we need to use integration by parts. Usually when we have something like $x^n e^x$, we let $u = x^n$ and $dv = e^x\,dx$, so that we’re decreasing the exponent in the $x^n$ term when we take $\int vdu$. Here, since we have $e(x^2)$, we need to have $dv = xe(x^2)\,dx$ in order to be able to find $v = \int dv$, so this leaves $x^2$ for $u$. That is, $x^3e(x^2)$ can be considered as $(x^2)(xe(x^2))$. Letting $u = x^2$ and $dv = xe(x^2)\,dx$ we get $du = 2x\,dx$ and $v = \frac{e(x^2)}{2}$. Thus we have:

$$
\int x^3e(x^2)\,dx = x^2 \left( \frac{e(x^2)}{2} \right) - \int \left( \frac{e(x^2)}{2} \right) (2x)\,dx
$$

$$
= \frac{x^2 e(x^2)}{2} - \int xe(x^2)\,dx
$$

$$
= \frac{x^2 e(x^2)}{2} - \frac{e(x^2)}{2} + C
$$

Notice: Here, evaluating $\int vdu$ is just a repetition of when we evaluated $v = \int dv$, since it turns out that $vdu = dv$.

Using this result in the formula for the general solution found previously, we get:

$$
y = \frac{1}{e(x^2)} \int x^3e(x^2)\,dx
$$

$$
= \frac{1}{e(x^2)} \left( \frac{x^2 e(x^2)}{2} - \frac{e(x^2)}{2} + C \right)
$$

$$
= \frac{x^2}{2} - \frac{1}{2} + \frac{C}{e(x^2)}
$$

Now, using the side condition $y(1) = 0$, we get:

$$
y(1) = 0 \Rightarrow \left[ \frac{x^2}{2} - \frac{1}{2} + \frac{C}{e(x^2)} \right]_{x=1} = 0
$$

$$
\Rightarrow \frac{1}{2} - \frac{1}{2} + \frac{C}{(1)^2} = 0
$$

$$
\Rightarrow 0 + \frac{C}{e} = 0
$$

so we see that $C = 0$. Thus, the specific function we are looking for is $y = \frac{x^2}{2} - \frac{1}{2}$. 

12
Exponential Growth and Decay

We now introduce a subtopic of differential equations, **exponential growth and decay**. Exponential functions are very useful for modelling many real world phenomena. Exponential functions come into play in situations in which the rate at which some quantity grows or decays (i.e., increases or decreases over time) is proportional to the quantity present. Letting $y$ denote the quantity which is changing over time, we have that the change in $y$, $\frac{dy}{dt}$, is proportional to $y$, so we get a differential equation of the form $\frac{dy}{dt} = ky$ for some constant $k$. This is a separable D. E., so we could use our usual method. However, we can take advantage of the specific form to get a short cut method for solving these problems.

Consider the steps we would follow in solving a differential equation of this form. We have $\frac{dy}{dt} = ky$. Provided that $y \neq 0$, we can separate this to $\frac{1}{y}dy = kdt$, which gives:

$$
\int \frac{1}{y}dy = \int kdt
\Rightarrow \ln |y| = kt + C
\Rightarrow e^{\ln |y|} = e^{kt+C}
\Rightarrow |y| = e^{kt}e^C
\Rightarrow y = \pm e^C e^{kt}
\Rightarrow y = Ae^{kt}
$$

In the last line above, we have substituted the arbitrary constant $A$ for the messier arbitrary constant $\pm e^C$, so $A$ can have any value but 0. Now, if we know the initial quantity present, $y(0) = y_0$ for some value $y_0$, we have $y(0) = Ae^{k(0)} = Ae^0 = A$, so we get $A = y_0$. Thus, the specific solution to the D. E. $\frac{dy}{dt} = ky$ is $y = y_0e^{kt}$.

We will get this same form for any D. E. of this type, so we do not need to carry through these steps each time. Instead, we use the following theorem (which we have just proved).
Theorem 10.8. The solution to the differential equation \( \frac{dy}{dt} = ky \), subject to \( y = y_0 \) at \( t = 0 \) is given by

\[ y = y_0 e^{kt} \]

That is, if we know that the initial quantity, at time \( t = 0 \), is \( y_0 \) and that the rate of change is \( \frac{dy}{dt} = ky \), then we can always find an expression for the quantity present at time \( t \) as \( y = y_0 e^{kt} \).

We look at 3 applications of exponential growth and decay.

Biological Growth

Example 12. Suppose it is known that the cells of a given bacterial culture divide every 3.5 hours (on average). If there are 500 cells in a dish to begin with, how many will there be after 12 hours?

Solution: Here, we have growth arising from cell division. We are told that the cells divide every 3.5 hours, so the number of cells doubles every 3.5 hours. Thus, the rate of change in the number of cells present is proportional to the number of cells present. Recognizing that this is an exponential growth problem, and that we have \( y_0 = 500 \), we let \( y \) be the number of cells present at time \( t \) and use the theorem to see that

\[ y = y_0 e^{kt} = 500 e^{kt} \]

The only unknown left to determine is \( k \). The information provided that the bacterial culture divides every 3.5 hours allows us to find the value of \( k \). The number of cells doubles every 3.5 hours, so after 3.5 hours there should be 1000 cells. That is, at time \( t = 3.5 \), we have \( y = 1000 \). So we see that:

\[ 1000 = 500 e^{k(3.5)} \]

\[ 2 = e^{3.5k} \]

\[ \ln 2 = 3.5k \]

\[ k = \frac{\ln 2}{3.5} \approx 0.198042 \]

Substituting this into the solution formula \( y = y_0 e^{kt} \) we have:

\[ y = 500e^{(0.198042)t} \]

Therefore after 12 hours there will be \( 500e^{(0.198042)(12)} \approx 5383 \) cells in the dish.
Radioactive Decay

**Definition 10.9.** The *Half-life* of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay (i.e. for the quantity to be reduced by one-half).

The half-life of an element is totally independent of the number of nuclei present initially, because this decay occurs exponentially, i.e. according to the differential equation \( \frac{dy}{dt} = ky \), where \( k \) is some negative constant. (The value of the constant differs for the various radioactive elements.)

Let \( y_0 \) be the number of radioactive nuclei present initially. Then the number \( y \) of nuclei present at time \( t \) will be given by:

\[
y = y_0 e^{kt}
\]

Since we are looking for the half-life, we wish to know the time \( t \) at which only \( \frac{1}{2} \) \( (y_0) \) nuclei remain:

\[
y_0 e^{kt} = \frac{1}{2} y_0
\]

\[
\Rightarrow e^{kt} = \frac{1}{2}
\]

\[
\Rightarrow kt = \ln \left( \frac{1}{2} \right) = \ln 1 - \ln 2 = -\ln 2
\]

\[
\Rightarrow t = \frac{\ln 2}{-k}
\]

(Since \( k \) is negative, \( \frac{\ln 2}{-k} \) is positive.) Thus the half-life depends only on \( k \).

The formula above is worth noting for future use:

\[
\text{half-life} = \frac{\ln 2}{-k}
\]
Example 13. The number of atoms of plutonium-210 remaining after \( t \) days, with an initial amount of \( y_0 \) radioactive atoms, is given by:

\[
y = y_0 e^{(-4.95 \times 10^{-3})t}
\]

Find the half-life of plutonium-210.

Solution: We see that for this element we have \( k = -4.95 \times 10^{-3} \). Using the formula above we have:

\[
\text{Half-life} = \frac{\ln 2}{-k} = \frac{\ln 2}{4.95 \times 10^{-3}} \approx 140
\]

Therefore the half-life of plutonium-210 is approximately 140 days.

Example 14. Uranium 237 has a half-life of about 6.78 days. If there are 10 grams of Uranium 237 now, how much will be left after 2 weeks?

Solution: We have \( \frac{dy}{dt} = ky \) for some constant \( k \) and we know that there are \( y_0 = 10 \) grams present initially, so the amount left at time \( t \) is given by \( y = 10e^{kt} \). To find the value of \( k \), we use the half-life, i.e. the fact that half will be gone in about 6.78 days, so that \( y(6.78) = 5 \). We have:

\[
10e^{6.78k} = 5
\]

\[
\Rightarrow e^{6.78k} = \frac{1}{2}
\]

\[
\Rightarrow 6.78k = \ln \left( \frac{1}{2} \right) = -\ln 2
\]

\[
\Rightarrow k = \frac{-\ln 2}{6.78} \approx -0.1022
\]

Note: We could also have found this by rearranging the previously stated formula for half-life to solve for \( k \) in terms of the half-life as \( k = \frac{-\ln 2}{\text{half-life}} \).

Now, substituting for \( k \) in our general solution, \( y = 10e^{kt} \), we see that the function \( y = 10e^{-0.1022t} \) gives the approximate amount remaining after \( t \) days. Therefore, after 2 weeks (i.e. 14 days), the amount still remaining will be \( y(14) = 10e^{(-0.1022)(14)} \approx 2.39 \) grams.
Carbon Dating

One particular application of radioactive decay is its use in determining how long tissue which was once alive has been dead, using Carbon Dating. This is possible because Carbon 14 is a radioactive element which is present in all living tissue. While the tissue is alive, Carbon 14 is replaced by absorption from the atmosphere at the same rate at which it decays. By this means, the ratio of Carbon 14 to the non-radioactive (i.e. stable) Carbon 12 remains constant in the tissue. However, when the living tissue dies, it stops absorbing Carbon 14, and so the quantity of this element is not replenished as it decays, causing the ratio of Carbon 14 to Carbon 12 to decrease. That is, there is exponential decay of the Carbon 14, while the stable Carbon 12 does not decay and hence remains at a constant level. Thus, given a sample of tissue which was once alive, it is possible to discover how long ago it died by determining what proportion of its Carbon 14 is still present, using the fact that the half-life of Carbon 14 is about 5570 years.

**Example 15.** Find the age of an object that has been excavated and found to have 90% of its original amount of radioactive Carbon 14.

**Solution:** Using the equation $y = y_0e^{kt}$ we see that we must find two things:

(i) the value of $k$
(ii) the value of $t$ for which $y_0e^{kt} = \frac{90}{100}y_0$, i.e., find $t$ such that $e^{kt} = \frac{9}{10}$.

(i) **Find the value of $k$:**
Rearranging the half-life equation and using the fact that the half-life is known to be 5570 years, we have

$$-k = \frac{\ln 2}{\text{half-life}} = \frac{\ln 2}{5570} \approx .0001244$$

So $k = -.0001244$.

(ii) **Find the value of $t$ which makes $e^{kt} = \frac{9}{10}$:**

$$e^{-0.0001244t} = 0.9 \Rightarrow -.0001244t = \ln(0.9)$$

$$\Rightarrow t = \frac{\ln(0.9)}{-0.0001244} \approx 878$$

Therefore the sample is approximately 878 years old.
Mathematical Modelling

Definition 10.10. A mathematical model of a real-life situation is a set of mathematical statements which, when taken together, describe the situation in mathematical terms.

Example 16. The slope of the tangent line to a particular curve at any point \((x, y)\) on the curve is given by \(f(x, y)\). The curve passes through a known point \((x_1, y_1)\). Construct a mathematical model of this situation.

Solution: Let the curve be \(y = y(x)\) for some function \(y(x)\). Then the tangent lines to the curve have slope given by \(\frac{dy}{dx}\), so we have \(\frac{dy}{dx} = f(x, y)\). For \((x_1, y_1)\) to be a point on the curve, it must be true that \(y(x_1) = y_1\). Thus, we get the mathematical model:

\[
\frac{dy}{dx} = f(x, y) \\
y(x_1) = y_1
\]

For a situation such as in this example, when we know the function \(f(x, y)\) (which normally we would) and also the values of \(x_1\) and \(y_1\), we can solve the mathematical model to find the function \(y(x)\), as we have done previously (assuming, of course, that the D.E. is either linear or separable). Having solved a mathematical model, we can answer other questions about the situation. For instance, in this case we could determine a point on the curve with a particular \(x\)-coordinate, or a particular slope, etc.

Notice that there is nothing new happening here. We have constructed and then solved mathematical models many times, without saying that was what we were doing. Whenever we are given a “word problem” and translate it into mathematical terms, we are constructing a mathematical model. In the current section we will be constructing and solving mathematical models for several kinds of real-world problems in which first order differential equations arise.

We look at three main types of problem involving the applications of first order differential equations: (i) Heating and Cooling problems; (ii) Mixing problems; and (iii) Falling Object problems. We will discuss and look at an example of each type of problem.
Heating and Cooling Problems

For this type of problem, the question involves the temperature \( T \) of a certain object, which is placed in a medium of constant temperature \( M \) so that as time \( t \) varies, so does \( T \), (so \( T \) has a rate of change with respect to \( t \)). In this case, Newton’s law of cooling tells us the following:

\[
\frac{dT}{dt} = k(T - M) \text{ for some constant } k
\]

Another way of saying this is that the rate of change, over time, of the temperature of the object is proportional to the difference between the temperature of the object and that of the medium in which it is placed. Examples of heating and cooling problems include an ice cube tray of water put into a freezer, a cup of coffee set down in a room, or a roast put into a hot oven.

For this kind of problem, we generally know:

1. the initial temperature of the object, \( T_0 \)
2. the constant temperature of the medium, \( M \), and
3. some other observation of the temperature of the object, at some known time, i.e. that at time \( t_1 \), the temperature was \( T_1 \).

These pieces of information, together with Newton’s law of cooling, give us the mathematical model:

\[
\frac{dT}{dt} = k(T - M)
\]

\[
T(0) = T_0
\]

\[
T(t_1) = T_1
\]

We can solve this model to get \( T(t) \), a specific formula giving the temperature of the object at time \( t \). Once we have this formula, we can answer any other questions of interest.

\textit{Example 17.} A steak is removed from a freezer and put into the refrigerator to thaw. The freezer is kept at \(-10^\circ C\) and the fridge is kept at \(4^\circ C\). After 4 hours, the temperature of the steak was \(-6^\circ C\). When will the steak be thawed to \(2^\circ C\)?
Solution: We have the medium the steak is put into being the air in the refrigerator, so we have \( M = 4 \). The initial temperature of the steak is the temperature of the freezer, giving \( T_0 = -10 \). We let \( T = T(t) \) be the temperature of the steak \( t \) hours after it is placed in the fridge. Using all of this information, we get the mathematical model:

\[
\frac{dT}{dt} = k(T - 4)
\]

\[
T(0) = -10
\]

\[
T(4) = -6
\]

Notice: What we have is a separable differential equation:

\[
\frac{dT}{dt} = k(T - 4) \Rightarrow \frac{1}{T - 4}dT = kdt
\]

\[
\Rightarrow \int \frac{1}{T - 4}dt = \int kdt \Rightarrow \ln|T - 4| = kt + C
\]

At this point in a heating or cooling problem, we can use our knowledge of the situation to get rid of the absolute value signs. In this problem, the steak is colder than the medium, so the object here is heating, not cooling. That is, \( T < M \), so we have \( T - M < 0 \) in this instance. Therefore \( |T - M| = -(T - M) = M - T \). If we had a situation in which an object was cooling, we would have \( T > M \) so that \( T - M > 0 \) and \( |T - M| = T - M \).

Using \( |T - 4| = 4 - T \) in this case, we have:

\[
\ln(4 - T) = kt + C
\]

\[
\Rightarrow 4 - T = e^{kt+C} = e^{kt}e^C = Ae^{kt}
\]

\[
\Rightarrow T = 4 - Ae^{kt}
\]

Now, we know that \( T(0) = -10 \), so we have:

\[
-10 = 4 - Ae^{0k} = 4 - A \Rightarrow A = 14
\]

Therefore we see that

\[
T = 4 - 14e^{kt}
\]

We still need to find the value of \( k \). This is what we use the other observation of temperature for. We are told that \( T(4) = -6 \).
This gives:

\[-6 = 4 - 14e^{4k} \Rightarrow 14e^{4k} = 10 \Rightarrow e^{4k} = \frac{10}{14}\]

\[\Rightarrow 4k = \ln\left(\frac{10}{14}\right) \Rightarrow k = \frac{\ln\left(\frac{5}{7}\right)}{4} \approx -0.08412\]

This gives the temperature function as \(T = 4 - 14e^{-0.08412t}\).

**Notice:** We know that for an object that is heating, the medium is warmer than the object so that the difference \(T - M\) is negative. Also, since the object is heating, the change in temperature over time is positive, so we must have \(\frac{dT}{dt} > 0\). Here, we found that \(k\) was negative, which does in fact give \(\frac{dT}{dt} = k(T - M) > 0\) as required. That is, \(\frac{dT}{dt}\) is the product of 2 negatives, and hence is positive, as expected.

Now that we have found the formula for \(T\), we can answer the question. We want to know when the steak will reach 2°C.

We have:

\[2 = 4 - 14e^{-0.08412t} \Rightarrow 14e^{-0.08412t} = 4 - 2 = 2\]

\[\Rightarrow e^{-0.08412t} = \frac{2}{14} = \frac{1}{7}\]

\[\Rightarrow -0.08412t = \ln\left(\frac{1}{7}\right)\]

\[\Rightarrow t = \frac{\ln\left(\frac{1}{7}\right)}{-0.08412} \approx 23.13\]

Since we are measuring \(t\) in hours, we see that it will take almost a full day for the steak to thaw.

**Notice:** In these problems, the object never, of course, passes the temperature of the medium. We can see in this example that as \(t\) gets large, \(e^{-0.08412t}\) gets close to 0 so that \(T(t) = 4 - 14e^{-0.08412t}\) gets very close to (but never exceeds) 4.
Mixing Problems

All of the mixing problems that we will be dealing with will involve a “tank” into which a certain substance will be added at a certain input rate (measured in kg/hr or lb/min or some such units), and the substance will leave the system at a certain output rate (measured in the same units). We shall always use $y = y(t)$ to denote the amount of the substance in the tank at time $t$. Figure 1 depicts such a tank.

![Diagram of a mixing tank](image)

Figure 1: Mixing Problem

**Note:** The “substance” we are interested in in these problems is generally contained in, and/or being mixed with, some other medium (for instance, water). Thus, the concentration of the substance in the medium when it enters the tank is generally not the same as the concentration of the substance in the medium when it leaves the tank. It is important to be clear on the fact that it is the input rate and output rate of the substance which are needed, and that $y = y(t)$ is measuring the amount of the substance, not the volume of the whole mixture, which is present in the tank at time $t$.

Often, what we know is the inflow rate of a mixture containing the substance, and the outflow rate at which the mixture is leaving the tank. Knowing the inflow rate and the concentration of the substance in the mixture flowing in allows us to determine the input rate at which the substance itself is entering the tank. We generally assume that mixing within the tank takes place instantaneously, so that the outflow concentration is simply the quantity of the substance present in the tank, divided by the total volume of the mixture in the tank. This, together with the outflow rate, allows us to
determine the output rate.

The differential equation involved in this kind of problem arises from the following natural relationship – that the rate at which the amount of the substance in the tank is changing (over time) is simply the difference between the rate at which it is coming into the tank and the rate at which it is leaving the tank. That is, we have:

\[
\frac{dy}{dt} = (\text{input rate}) - (\text{output rate})
\]

For this kind of problem, we generally know:

1. the volume in the tank initially,
2. the (constant) inflow rate and concentration of the inflow,
3. the (constant) outflow rate, and
4. the quantity of the substance which is present in the tank initially, \(y_0\).

We will only be looking at situations in which the inflow and outflow rates are equal, so that the total volume in the tank is constant (although the concentration of the substance changes over time).

The mathematical model for this kind of situation is therefore:

\[
\frac{dy}{dt} = (\text{input rate}) - (\text{output rate})
\]

\[
y(0) = y_0
\]

where the input rate is generally a constant, and the output rate is a function of \(y\), the quantity present. We solve this model to find \(y(t)\), the function giving the quantity of the substance present in the tank at time \(t\).

Example 18. Initially, a water tower contains 1 million litres of pure water. Two valves are then opened. One valve allows a solution of water and fluoride, with a concentration of 0.1 kg of fluoride per litre of water, to flow into the tower at a rate of 80 litres per minute. The other valve allows the solution in the tank to be drained at 80 litres per minute. Assume that the solution is mixed constantly, so that we have a homogeneous fluid in the tank, i.e., at any point in time, the concentration of fluoride in the water is uniform throughout the tank.
(a) Find an expression for the amount (in kg) of fluoride in the water tower $t$ minutes after the valves are opened.
(b) Determine how long it will take for the concentration of fluoride in the water to reach .05 kg/l.

Figure 2:

Solution:
(a) Let $y = y(t)$ be the number of kilograms of fluoride in the tank at time $t$. We need to find both the input and output rates. Since water (with fluoride) is flowing in and out at 80 l/min, the volume in the tank remains constant at 1,000,000 l.

**Input Rate:** We have 80 litres per minute entering the tank, with each litre containing 0.1 kg of fluoride, so each minute we have $(80)(0.1) = 8$ kg of fluoride entering the tank. Thus, we have: **Input Rate = 8 kg/min.**

**Output Rate:** The entire system contains $y$ kg of fluoride at any given time, distributed throughout the one million litres in the tank. This means that each litre of water in the tank contains $\frac{y}{1,000,000}$ kg of fluoride at any given moment in time. We have 80 litres per minute leaving the system, therefore in any one minute we have $(80)\left(\frac{y}{1,000,000}\right)$ kg of fluoride leaving the tank. That is, we have: **Output Rate = $(8)\left(\frac{y}{1,000,000}\right)$ kg/min.**
We use the differential equation \( \frac{dy}{dt} = (\text{input rate}) - (\text{output rate}) \). Substituting the rates above into this formula, we get:

\[
\frac{dy}{dt} = 8 - (8)\left(\frac{y}{100000}\right)
\]

In order to solve this differential equation, we recognize it as a linear D.E.:

\[
\frac{dy}{dt} + 0.00008y = 8
\]

Recall that given a linear differential equation, \( \frac{dy}{dt} + p(t)y = q(t) \), we use the integrating factor \( \Phi(t) = e^{\int p(t)dt} \) and then the general solution is given by:

\[
y = \frac{1}{\Phi(t)} \int q(t)\Phi(t)dt
\]

Here, we have \( p(t) = 0.00008 \), and \( q(t) = 8 \), so we get the integrating factor \( \Phi(t) = e^{\int 0.00008dt} = e^{0.00008t} \). Substituting \( \Phi(t) \) and \( q(t) \) into the above formula, we get:

\[
y = e^{-0.00008t} \int 8e^{0.00008t}dt
\]

\[
= e^{-0.00008t} \left( \frac{8e^{0.00008t}}{0.00008} + C \right)
\]

\[
= e^{-0.00008t} \left( 100000e^{0.00008t} + C \right)
\]

\[
= 100000 + Ce^{-0.00008t}
\]

That is, the general solution to the D.E. is \( y = 100,000 + Ce^{-0.00008t} \). We need to find the specific solution, by solving for \( C \). We are told that initially, the tower held pure water. That is, at time \( t = 0 \) there was no fluoride in the tank, so we have the side condition \( y(0) = 0 \). Substituting this into the general solution, we get:

\[
0 = 100000 + Ce^0 \Rightarrow C = -100000
\]

We substitute this into the general solution to get the specific solution:

\[
y = 100000 - 100000e^{-0.00008t}
\]

(b) We need to determine how long it will take for the concentration to reach a level of \( .05 \) kg/l. This concentration of fluoride in the 1,000,000 l in the tank corresponds to having \( (1000000)(.05) = 50000 \) kg of fluoride in total in the tank. Thus we see that we need to find the value of \( t \) that satisfies \( y = 50000 \). We do this using the formula we found in part (a).
50000 = 100000 - 100000e^{-0.00008t}
⇒ \frac{1}{2} = 1 - e^{-0.00008t}
⇒ e^{-0.00008t} = \frac{1}{2}
⇒ \ln(e^{-0.00008t}) = \ln\left(\frac{1}{2}\right)
⇒ -0.00008t = -\ln 2
⇒ t = \frac{\ln 2}{0.00008}
≈ 8664.34

We see that it will take approximately 8664.34 minutes ≈ 144.41 hours, or just over 6 days, for the concentration of fluoride to reach .05 kg/l.

In more complicated mixing problems, the total volume in the tank may itself be changing over time, if the inflow rate of the substance or mixture is not the same as the outflow rate of the mixture. Solution of such a problem follows the same basic steps, but is slightly more complicated. In that kind of situation, the volume in the tank is a function of $t$, so the output rate involves both $y$ and $t$.

**Falling Object Problems**

In these problems, we start with a standing object which is dropped from height $y_0$ at time $0$. That is, the object is not moving before time $t = 0$, and is not pushed with any force, but merely dropped. Gravity causes the object to fall and to accelerate as it falls. We will make the simplifying assumption that air resistance is negligible, and so can be ignored. We use the fact that the acceleration due to the downward force of gravity is $g = -9.81 m/s^2 = -32 ft/s^2$. (Note: The acceleration is considered to be negative because the motion of the object is downward.)

Let $y = y(t)$ be the height of the object at time $t$. Let $a(t)$ be the acceleration, and $v(t)$ be the velocity of the object, at time $t$. We use the following facts:
1. acceleration is the rate of change in velocity over time, so \( a = \frac{dv}{dt} \).

2. velocity is the rate of change in position over time, so \( v = \frac{dy}{dt} \).

Since, with no air resistance, the acceleration of the object is constant at \( g \), we get the mathematical model:

\[
\frac{dv}{dt} = g \quad \frac{dy}{dt} = v \quad v(0) = 0 \quad y(0) = y_0
\]

**Example 19.** An object is dropped from a height of 500 m. When will the object reach ground level, and with what speed?

**Solution:** We see that this is a falling object problem in which the units are metric, so we use \( g = -9.81 \text{m/s}^2 \). We let \( y = y(t) \) be the height of the object above ground level, in metres, at time \( t \). We are told that the object is dropped from a height of 500 m, so we know that when \( t = 0 \), \( y = 500 \). Thus the mathematical model is:

\[
\frac{dv}{dt} = -9.81 \quad \frac{dy}{dt} = v \quad v(0) = 0 \quad y(0) = 500
\]

Of course, we have:

\[
\frac{dv}{dt} = -9.81 \Rightarrow dv = -9.81 dt \Rightarrow v = -9.81 t + C
\]

Hence, \( v(0) = 0 \) gives \(-9.81(0) + C = 0 \), so we have \( C = 0 \) and \( v = -9.81 t \).

And since we have \( \frac{dy}{dt} = v \), we get:

\[
\frac{dy}{dt} = -9.81 t
\]

\[
\Rightarrow dy = (-9.81 t) dt
\]

\[
\Rightarrow y = -9.81 \left( \frac{t^2}{2} \right) + C
\]

\[
\Rightarrow y = -4.905 t^2 + C
\]

Using the fact that \( y(0) = 500 \), we get:

\[
500 = -4.905(0)^2 + C \Rightarrow C = 500
\]

Therefore the function which gives the height of the object at time \( t \) is:

\[
y = -4.905 t^2 + 500
\]
Now, we need to find the time at which the object hits the ground, i.e. find $t$ for which $y = 0$. Letting $y = 0$ in the last equation, we get:

$$0 = -4.905t^2 + 500 \Rightarrow t^2 = \frac{500}{4.905} \approx 102 \Rightarrow t \approx \sqrt{102} \approx 10.1$$

We see that the object reaches ground level approximately 10.1 seconds after it begins to fall.

To find the speed at which the object is travelling when it reaches ground level, we simply use the fact that speed is the absolute value of velocity, together with the formula for $v$ we found along the way, i.e. $v = -9.81t$, and the fact that the object reaches the ground at time $t = 10.1$.

At $t = 10.1$ we have $v = -9.81(10.1) \approx -99.1 \text{ m/s}$. Therefore the speed of the object is about 99.1 m/s when it reaches ground level.